THEORY OF

Steel-Concrete Arches

AND OF

Vaulted Structures.

BY

WILLIAM CAIN,
MEMBER AM. SOC. C. E.
PROF. OF MATHEMATICS, UNIVERSITY OF NORTH CAROLINA.

SECOND EDITION, THOROUGHLY REVISED.

NEW YORK:
D. VAN NOSTRAND COMPANY.
23 MURRAY AND 27 WARREN STREETS.
1902.

PREFACE.

THE author has availed himself of the opportunity, in this revision, to give a complete solution of the elastic arch of variable section. The arch of steel and concrete combined is taken up in detail to illustrate the general graphical treatment. The method to follow for other arches, as those of steel, stone or concrete, is at once apparent from the general solution.

The aim throughout has been to give a clear analysis of principles involved, a knowledge of the fundamental principles of the equilibrium polygon being alone assumed.

This book, although independent of the author's other works on Arches, really supplements them, since in "Theory of Voussoir Arches," the elastic arch of constant section is alone treated, and in "Theory of Solid and Braced Elastic Arches," the arch of variable section is very briefly considered without a completely worked out example.

This work then deals especially with the most complicated case, and it is believed that a thoroughly practical solution is offered.

About one half of this second edition is entirely new; the remainder is substantially as it appeared in the first edition,* though often more condensed.

Credit is given in the body of the work to those from whom the author has derived assistance. Special mention though is due Dr. Hermann Scheffler, as his "Théorie des Voutes" has been drawn upon for much of the theory in treating culverts and vaulted structures.

The theory of the elastic arch for arch *bridges* of stone, concrete, brick, etc., is advocated here, as in the author's previous treatises. This theory, as a practical working theory, receives direct confirmation in the many experiments performed by the Austrian Society of Civil Engineers and Architects, in 1890–95, on arches of brick and concrete.

CHAPEL HILL, N. C., August 11, 1901.

* The title of previous edition was "Voussoir Arches Applied to Stone Bridges, Tunnels, Domes, and Groined Arches."

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THEORY OF THE ARCH.

CHAPTER I.

ARCHES OF VARIABLE SECTION UNDER VERTICAL LOAD.

INTRODUCTORY.

1. The theory of the arch with variable section is considered in what follows, and a practical solution is offered that is applicable to arch bridges of any material, such as steel, concrete, or steel and concrete combined.

To secure the greatest generality, the solid concrete arch of variable section, with embedded steel ribs, is first fully treated. The modification necessary when the ribs are omitted is at once apparent, in which case the theory applies to solid arches of any material, stone, steel, concrete, etc., of variable section, and, in

fact, to the "braced arch" of any shape or design. The theory of the arch with constant section follows as a special case.

Therefore, although the steel-concrete arch is alone illustrated in the worked out examples given, the treatment is intended to show the working of the method to arches of any kind, and the method to follow in any case is fully indicated in the text.

Of the various types of arch bridges, those in steel and concrete combined, when scientifically constructed, possess many advantages, in that they can be easily moulded to the most desirable and beautiful forms, and the embedded steel ribs can supply, when needed, tensile resistances, which arches in stone, brick or concrete alone are not so well calculated to furnish. The steel, too, is protected from the air by the concrete, so that the structure should last indefinitely. It thus presents a marked contrast to the unsightly steel truss with parallel chords, which counts its life in decades. The concrete has not the com-

pressive strength of stone, but for ordinary spans, this is not a matter of importance, as the increase of section in going from the crown to the abutment, required to avoid too much tension, is generally more than sufficient to provide for compression.

It is perfectly practicable to construct an arch of steel ribs braced vertically, with stone voussoirs between the flanges. The combination would furnish all needed tensile and compressive strength for even large spans; but it would be difficult to protect the steel from rusting, perhaps, even with a concrete coating. The main objection to it, however, is that stone has not the same co-efficient of expansion from heat as steel. Certain authorities state that concrete *has*, and further, experiment shows that concrete and steel (free from rust, paint, scale and oil) exhibits a large adhesive strength. The combination then possesses so many good qualities that it will be thoroughly treated in what follows.

FORMULAS FOR UNIT STRESS FOR A STEEL CONCRETE ARCH.

2. Let us consider a concrete arch with steel bars embedded in the concrete as shown by the dotted lines in the longitudinal section, Fig. 1, and in cross section by the little rectangles in Fig. 2, repre-

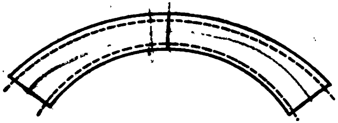

Fig. 1.

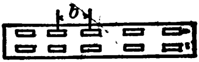

Fig. 2.

senting a cross section of the arch anywhere. The pair of bars in the same vertical plane will be called a *rib*, and they may be of any pattern (angles, plates, etc.), and connected by latticing if preferred, though the latter is not necessary, since the concrete is fully capable of taking the

shearing forces (exerted at right angles to the ribs) which are small. It is understood that the steel bars are free of paint, oil, scale or rust, so that when embedded, the adhesion between the steel and concrete will be complete, and sufficiently great to cause the concrete and steel to act as one mass. As Prof. Bauschinger found the adhesion between iron and concrete to be from 570 to 640 pounds per square inch, which is about double the tensile strength of concrete, there should be no difficulty in having the assumption realized.

3. The steel ribs are generally spaced uniformly a few feet apart, and in consequence a very rough approximation has to be resorted to in practice to apply theory to the really very complicated case.

It is assumed, that it is approximately correct to consider the material of the upper bars to be distributed uniformly along an arch sheet that passes through the centre lines of the upper bars, and that the material of the lower bars is similarly dis-

tributed uniformly along an arch sheet that passes through the centre lines of the lower bars; so that if A = area of cross section of the two bars constituting a rib, in square feet, and if the ribs, are spaced b feet apart, then $A \div b = A_2$ will be the area in square feet of the cross section of steel supposed in a slice of the arch contained between two vertical longitudinal planes one foot apart. It would be safer, perhaps, to allow only a fraction of A_2 in the computation, and certainly it would seem advisable to limit b to a certain maximum, but as this must remain for the present, a matter of judgment, the simple assumption above will be made in what follows, and A_2 in the formulas can be altered to suit the judgment of the engineer. Therefore a longitudinal slice of the arch contained between two vertical planes one foot apart, will be assumed to have the cross-section, Fig. 3.

In this cross-section, which is properly taken perpendicular to the neutral surface

and approximately at right angles to the soffit,

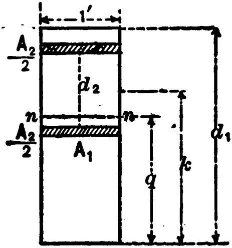

Fig. 3.

A_1 = area of concrete in square feet.
A_2 = area (shaded) of two steel bars in square feet = $A \div b$.
d_1 = depth of arch in feet.
d_2 = depth of steel rib in feet.
k = distance in feet from soffit to centre of gravity of steel rib.

The modulus of elasticity in pounds per square foot,

for concrete = E_1,
for steel = $nE_1 = E_2$.

The neutral surface of the arch, where

the stress due to bending moments *only* is zero, intersects the plane of the cross-section, Fig. 3, in *nn*.

4. In Fig. 4 is shown a side view of a part of the supposed arch, 1 foot wide (of

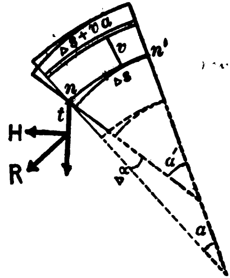

Fig. 4.

which Fig. 3 is the cross section), contained between two planes perpendicular to the neutral surface nn', and making an angle α in circular measure, before strain between them. A vertical plane midway between the faces of the supposed arch, intersects the neutral surface in the line $nn^1 = \Delta s$ feet in length, which may be called the neutral line, and the forces act-

ing upon the artificial voussoir considered, will have their resultants acting in this medial plane; hence the problem is referred to one of forces acting in one plane.

Let R be the resultant of all external forces acting upon the section passing through n, the forces considered being the right reaction, and all loads acting on the arch from the right abutment up to the section (or joint) passing through n. As is well known, when the true equilibrium polygon has been located, the line of action of R is given by the side of the equilibrium polygon pertaining to n, and its amount and direction, from the ray of the force diagram, parallel to this side.

As α and Δs will be considered very small, the voussoir can be considered without weight or loads acting on it, and be treated as a free body acted on by R and the stresses resulting from the action of the part of the arch to the left of n upon the section through n. To ascertain these stresses, conceive applied at n two opposed forces + R, − R, each equal and

parallel to R. The single force R is thus replaced by a couple $R\overline{R}$ and a force $+R$ acting at n. The latter may be decomposed into components T and N tangential and normal to nn' at n. The force T causes a uniform shortening of the fibres on the voussoir (as will be proved in Art. 10) so that the section is moved parallel to itself under its action. This is not shown in Fig. 4, to avoid confusion. The force N acting along the section is the shearing force and having but a small effect in the deformation of the arch, is neglected.

5. The couple $R\overline{R}$ is principally effective in changing the curvature of the arch, and its moment is most conveniently found by multiplying the horizontal component of $R = H$, the pole distance, by the vertical distance from n to R. Thus call this distance in feet $= t$; then if R is resolved where the vertical through n cuts it, into a horizontal component H and a vertical component, the latter acts through

n; hence the moment M of R about $n =$ moment of couple $R\overline{R} = Ht$.

$$\therefore \quad M = Ht \; (\textit{in foot pounds}), \qquad (1)$$

when H is in pounds and t in feet.

Under the action of this couple, the angle α is changed to α', and the curvature is increased if R cuts the section below n (as then the greatest compression is at the intrados), and decreased when R cuts the section above n. If we call $\Delta\alpha = \alpha' - \alpha$, and regard M as + when right-handed, this amounts to saying that,

$\Delta\alpha$ is plus when M is plus and therefore R acts below n,

$\Delta\alpha$ is minus when M is minus and consequently R acts above n.

6. Call the distance of any fibre from $nn' = v$, this being plus for a fibre above nn', minus below. As nn' is very small, it can be treated as an arc of a circle, and the axis of a fibre in the same plane as concentric with it. The length of the fibre before flexure is $(\Delta s + v\alpha)$,

after flexure $(\Delta s + v\alpha')$; its change of length is, $v(\alpha' - \alpha) = v.\Delta\alpha$. Calling its cross-section $= a$ in square feet, and the unit stress on it due to $M = f$ in pounds per square foot, the stress on the fibre, if of concrete, is

$$fa = \frac{v.\Delta\alpha}{\Delta s + v\alpha} aE_1, \tag{2}$$

and if of steel,

$$fa = \frac{v.\Delta\alpha}{\Delta s + v\alpha} anE, \tag{3}$$

by the ordinary formula connecting stress and deformation,

$$f = \frac{\text{elongation of fibre}}{\text{length of fibre}} \times E.$$

It is plain that $(\Delta s + v\alpha)$ in the denominators, can be replaced by Δs without appreciable error. The sum of all the stresses (due to flexure only) acting on the entire section at n is, therefore,

$$\Sigma(fa) = \frac{E_1\Delta\alpha}{\Delta s} \Sigma(va), \tag{4}$$

where a is to be replaced by na in the summation for the steel bars; which is the same as if at the distance of each steel bar from the neutral axis, n times the same area of concrete was taken.

Recurring to the free body, Fig. 4, in equilibrium under the action of R and the resisting stresses along section n; since the algebraic sum of the components of all these forces perpendicular to section at n equals zero (by a law of mechanics), and since T, the component of R, is directly balanced by the stresses that cause a uniform shortening of the fibres on the section (Art. 10), it follows that the remaining normal stresses (due to M) must balance independently,

$$\therefore \quad \Sigma fa = o, \text{ or, } \Sigma va = o\,;$$

which shows that the neutral axis nn' passes through the centre of gravity of the revised area of the section.

Therefore, recurring to Fig. 3, with the notation of Art. 3, and calling q the distance in feet from the lower edge of

cross-section to the neutral line nn, we have, for the revised area,

$$q(A_1 + nA_2) = (A_1 \frac{d_1}{2} + nA_2 k) \qquad (5)$$

which determines q.

7. The moment of the stress (af) about n on any fibre is (afv),

$$\therefore \quad M = \Sigma(afv) = E_1 \frac{\Delta\alpha}{\Delta s} \Sigma(v^2 a),$$

for the revised area.

Designating by I_1 the moment of inertia of the concrete (of area A_1), and by I_2 the moment of inertia of the actual area of steel, A_2 (Fig. 3), both in feet,

$$\Sigma v^2 a = \Sigma(v^2 a) \text{ for concrete} + \Sigma(v^2 an) \text{ for steel} = I_1 + nI_2.$$

$$\therefore \qquad M = E_1 \frac{\Delta\alpha}{\Delta s} (I_1 + nI_2).$$

$$\therefore \qquad \Delta\alpha = \frac{M\Delta s}{E_1(I_1 + nI_2)}. \qquad (6)$$

By reference to Fig. 4 and noting that the sections were drawn perpendicular to

the neutral axis, it is seen that (6) gives $\Delta\alpha$, or *the change* in the angle between the tangents to the neutral line at n and n' due to the couple $\overline{\text{RR}}$ whose moment is M. This approaches the exact truth as near as we please, as $\Delta\alpha$ and Δs approach zero indefinitely. Therefore, if we could proceed by analysis alone, $\Delta\alpha$ and Δs are replaced by $d\alpha$ and ds in (6), and the result integrated to find the change in the inclination of the end tangents of the neutral line corresponding to a length s measured along that arc.

8. To apply the graphical method, however, an approximation must be introduced here whose significance must be carefully noted. The assumption is, that for an appreciable length of Δs (several feet, for instance) $\Delta\alpha$ is given by (6), provided M is taken as constant and equal to the value corresponding to the mid-point of nn', Fig. 4, or $\frac{1}{2}\,\Delta s$ distant from either n or n^1, E_1, I_1 and I_2, being likewise taken there.

As the total change in the inclination

of the end tangents for a length s is the sum of all the infinitesimal changes for the part of the arch considered, or,

$$\Sigma\left(\frac{\Delta s}{E_1(I_1 + nI_2)} M\right),$$

Δs being very small, the assumption above is that this expression is equal approximately to

$$\theta = \frac{s}{E_1(I_1 + nI_2)} M_0,$$

where $s = \Sigma(\Delta s)$, M_0 is the moment at the middle of s, and E_1, I_1 and I_2 the corresponding quantities at the same point.

This assumption, though not exact, is the most reasonable that can be made, but it can only be tested for I_1 and I_2 variable, in a numerical example. If E_1, I_1, I_2 are constant, as for an arch ring of the same cross-section and E_1 constant throughout, and Δs is also constant, then the previous supposed equality reduces to,

$$\Sigma(M.\Delta s) = M_0 s\ ;$$

so that, if s is laid off along a line and divided into the equal lengths Δs and ordinates M are laid off (say) at the middle of each Δs, then $\sum(M\Delta s)$ represents an area and M_0 is its mean ordinate. Now, as M is proportional to t (§ 5) and calling t_0 the value of t corresponding to M_0, if the true equilibrium curve for the arch pertaining to the space s considered satisfies the condition $\sum(t\,\Delta s) = t_0 s$, then the assumption is exactly realized. This will be nearly true if the successive values of t are equal or are increasing in going from one end of s to the other, or decreasing over the same length, but not for the case where t increases over a part of s and decreases over the remainder, as a rule. The extreme limit of error is reached when t_0, the vertical distance from the middle of s on the neutral axis to the equilibrium curve, is greater than any of the other t's over the same length s. This cannot be guarded against in advance, but a study of equilibrium curves shows that it can generally happen on only two divisions of the neutral line and even here, the error is often slight in consequence of the curve running nearly parallel to the neutral line for a good part of a usual division.

9. Let f_1 = stress per square foot on an extreme fibre of the concrete, whose dis-

tance from the neutral axis is v_1 feet; then from (2), $f_1 = v_1 E_1 \frac{\Delta\alpha}{\Delta s}$, and eliminating $\frac{\Delta\alpha}{\Delta s}$ between this equation and (6), we find,

$$f_1 = \frac{Mv_1}{I_1 + nI_2}.$$

Similarly the stress per square foot in an extreme fibre of the steel $= f_2$, distant v_2 inches from the neutral axis, from (3) and (6), is,

$$f_2 = \frac{Mv_2 n}{I_1 + nI_2}.$$

These two stresses are due entirely to the couple whose moment is M.

10. The stresses corresponding to a supposed *uniform* shortening of the fibres along the cross-section at n, Fig. 4 (see § 4), and therefore not included in the bending stresses, have now to be evaluated. This uniform shortening entails a uniform compressive unit stress $= p$ on the

concrete acting on the area A_1 and a unit stress np on the steel (both in pounds per square foot) acting on an area A_2, as follows from the fundamental formula connecting stress and deformation, since we have assumed $E_1 = nE_2$. The resultant, $p(A_1 + nA_2)$, acts at a distance q' above the lower edge of the joint, Fig. 3, and taking moments about that edge,

$$q'p(A_1 + nA_2) = p(A_1\frac{d_1}{2} + nA_2k).$$

$\therefore$ By comparison with (5), $q' = q$, or the resultant of the stresses corresponding to the uniform shortening, acts at n and is exactly opposed by T acting at that point as assumed.

$$\therefore \qquad T = p(A_1 + nA_2),$$

from which $p = T \div (A_1 + nA_2)$

and $np = nT \div (A_1 + nA_2)$.

11. From the last two articles are derived the total unit stress s_1 in pounds per square foot, exerted on the concrete

at the upper or lower edges of the cross-section,

$$s_1 = \frac{T}{A_1 + nA_2} \pm \frac{Mv_1}{I_1 + nI_2}, \quad (7)$$

and the total unit stress, s_2, in pounds per square foot, experienced by the upper and lower bars of the steel,

$$s_2 = \left(\frac{T}{A_1 + nA_2} \pm \frac{Mv_2}{I_1 + nI_2}\right)n. \quad (8)^*$$

All dimensions in these formulas are in feet.

For sections symmetrical with respect to the neutral line, Fig. 3,

$$q = \frac{d_1}{2}, \quad v_1 = \frac{d_1}{2}, \quad v_2 = \frac{d_2}{2},$$

$$I_1 = \frac{d_1^{\ 3}}{12}, \quad I_2 = \tfrac{1}{4} A_2 d_2^{\ 2} = A_2 v_2^{\ 2}.$$

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* Mr. Edwin Thacher, M. Am. Soc. C. E., published the formulas (7) and (8) above in "Engineering News" for Sept. 21, 1899, without demonstration. His article may be referred to as an excellent presentation of the practical aspects of steel-concrete construction and as giving an account of the various systems that have been used, and closing with his own specifieation.

Therefore,

$$\left.\begin{aligned} s_1 &= \frac{T}{A_1+nA_2} \pm \frac{Mv_1}{\frac{1}{12}d_1^3+nA_2v_2^2} \\ s_2 &= \left(\frac{T}{A_1+nA_2} \pm \frac{Mv_2}{\frac{1}{12}d_1^3+nA_2v_2^2}\right)n \end{aligned}\right\} \quad (9)$$

CONDITIONS FOR EQUILIBRIUM FOR ARCH WITH NO HINGES.

12. The neutral line, for the case where the steel bars are symmetrically placed with respect to the centre line of the arch ring, is the centre line itself and can at once be laid off.

For an unsymmetrical cross-section, points in the neutral line must be found and laid off by aid of (5), and a curve traced through them.

In either case, let *a b c*, Fig. 5, represent the neutral line of an unstrained arch with fixed end tangents, and let *s* represent a length of the neutral line whose centre is at *b*. When the arch is loaded either with its own weight only or in addition with a live load, the neu-

tral line changes shape, and it was agreed in Art. 8, to regard the change in the inclination of the end tangents to the neutral arc s, as given by the formula,

$$\theta = \frac{Ms}{E_1(I_1 + nI_2)},$$

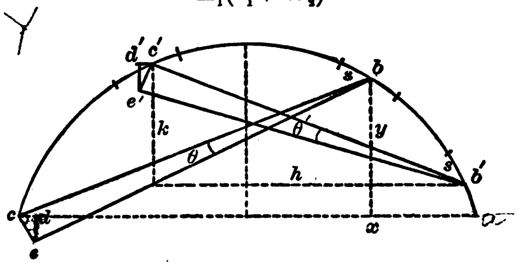

Fig. 5.

where n and E_1 are constant and M, I_1 and I_2 are taken at b, the middle of arc s.

Regard the end c temporarily free, then the bending in s alone will cause a rotation of the arc bc about b equal to θ, so that a line bc will rotate through an infinitesimal distance ce, taken as perpendicular to bc. Take c as origin, ca the axis of x and axis of y vertical and call

the co-ordinates of b, x and y. Draw ed perpendicular to ca; then from similarity of triangles,

$$cd = \frac{ce}{bc} y = y\,\theta, \qquad de = \frac{ce}{bc} x = x\,\theta.$$

The exact result would of course be found by dividing s into infinitesimal lengths Δs, and regarding the rotation to take place about the end of each little portion in turn, x and y thus having values above and below the means taken for b (the middle of s). We assume, therefore, in the graphical treatment to follow that if M, I_1, I_2, x, y are all taken at the mid-point of arc s, as a sort of average, that the horizontal and vertical deflections of c, due to s, are given nearly, by the above equations.

The total horizontal and vertical displacements of c due to the bending of all portions of the arch are then given by $\Sigma(y\,\theta)$, $\Sigma(x\,\theta)$, or

$$\Sigma \frac{Msy}{E_1(I_1 + nI_2)}, \qquad \Sigma \frac{Msx}{E_1(I_1 + nI_2)},$$

respectively, the summation including all the segments of the arch. The total *change* of inclination of tangents at a and c is similarly,

$$\Sigma\theta = \Sigma \frac{Ms}{E_1(I_1 + nI_2)}.$$

For an arch "fixed at the ends" and with no hinges, these three sums are zero; hence, we have the three conditions to be fulfilled, corresponding to end tangents fixed in direction, span invariable, and vertical deflection of c with respect to a, zero:

$$\Sigma \frac{Ms}{E_1(I_1 + nI_2)} = 0 \qquad (10)$$

$$\Sigma \frac{Mys}{E_1(I_1 + nI_2)} = 0 \qquad (11)$$

$$\Sigma \frac{Mxs}{E_1(I_1 + nI_2)} = 0. \qquad (12)$$

All dimensions being taken in feet and M expressed in foot pounds.

DEFLECTION AT THE CROWN.

13. It may be well to note here that if the actual moments M have been found for the mid-point of each small division of the neutral line, the horizontal and vertical displacements of any point as c' of the neutral line can be predicted. Thus, if b' is the centre of any division, calling h and k the horizontal and vertical projections of the chord $b'c'$ and θ' the angle (in circular measure) through which $b'c'$ rotates in consequence of the bending of that division, then the horizontal and vertical displacements of c' due to it are, respectively,

$$c'd' = \frac{c'e'}{b'c'}\,k = \theta' k, \qquad d'e' = \frac{c'e'}{b'c'}\,h = \theta' h,$$

and taking the algehraic sum of such quantities for all the divisions of the arc from a to c', the total displacements are found. Thus, inserting the value of θ, the total horizontal and vertical displacements of c' with respect to a, are

$$\Sigma\frac{\mathrm{M}sk}{\mathrm{E}_1(\mathrm{I}_1 + n\mathrm{I}_2)}, \qquad \Sigma\frac{\mathrm{M}sh}{\mathrm{E}_1(\mathrm{I}_1 + n\mathrm{I}_2)},$$

Note, from Art. 5, that M is plus when the bending is increased, or when R falls below the centre of the division b' considered, and M is

minus when the proper side of the equilibrium polygon (along which R acts) passes above b'.

In the first case, M plus, c' moves to the left and downward when c' is above b'; but when c' is below b' (as in the construction pertaining to b and c), then c' moves to the right and downwards. The reverse obtains when M is minus. Hence, if k is given the plus sign when c' is below b', the minus sign when c' is above b', the formulas above will give the horizontal movement of c', plus when to the right, minus when to the left, and the vertical displacement of c' plus downwards, minus when upwards.

When c' is at the crown k is always minus, and we have simply to attend to the signs of M, and it is evident that the displacements as found by the summation from a to the crown should equal those numerically found by summing from c to the crown.

This method, however attractive in theory will probably not work well in practice unless the true equilibrium polygon has been determined with great accuracy; for in a well designed arch, the moments $\mathrm{H}t = \mathrm{M}$, except at the springing joints, are small, or rather t is quite small, and a small absolute change in t often corresponds to a large percentage of its value, so that large errors can be made in the algebraic summations of the formulas.

The method has been tried for one bridge, however, and found to check satisfactorily with analytical solutions.

DIVISION OF THE NEUTRAL AXIS.

14. Recurring to the conditions (10), (11), (12) to be fulfilled by the equilibrium polygon for an arch "fixed at the ends," replacing M by Ht and regarding E_1 (the modulus for concrete) as constant throughout the arch ring, we observe, if the neutral line can be so divided that $s \div (I_1 + nI_2)$ is constant for each division, then $\dfrac{H}{E_1} \dfrac{s}{(I_1+nI_2)}$ can be placed before the sign of summation and the three conditions (10), (11), (12), reduce to,

$$\Sigma t = 0,\ \ \Sigma(tx) = 0,\ \ \Sigma(ty) = 0. \quad (13)$$

To show how to divide the neutral line so that $s \div (I_1+nI_2)$ shall be the same for each division, it is best to take a numerical example.

In the following table is given, for the

Thacher type of arch ring shown in Fig. 6 (plate), the radial depth d in feet, at l feet, measured along the neutral line, from the springing. The steel ribs, Figs. 1 and 2, are composed of two bars each, whose distances from the intrados and extrados are each 0.17 feet, and each bar has a gross section of $2\frac{5}{8} \times \frac{3}{4}$ inches. The splices in any one bar are to be made with $\frac{3}{4}$ inch rivets, so that the net section of any bar to resist tension is $\frac{15}{8} \times \frac{3}{4} = \frac{45}{32}$ square inches, or nearly $\frac{1}{100}$ square foot.

Assume the ribs 2 feet apart, centre to centre, and take $n = E_2 \div E_1 = 20$.

$$\therefore \quad A = \frac{2}{100}, \quad A_2 = \frac{1}{100}, \quad 20\,A_2 = 0.2.$$

$$I_1 + 20\,I_2 = \tfrac{1}{12}d^3 + 0.2\left(\frac{d}{2} - 0.17\right)^2$$

The neutral line, for this symmetrical section, coincides with the centre line of the arch ring and can be at once laid off. Call the successive lengths of the divisions measured along the neutral axis in

feet, beginning with the one next the springing, s_1, s_2, s_3, ..., respectively, and let d_1 be the radial depth of the arch ring at the middle of s_1 and d the depth at the middle of any division s. Then in order that $s \div (I_1 + nI_2)$ be constant,

$$\frac{s}{\frac{1}{12}d^3 + 0.2\left(\frac{d}{2} - 0.17\right)^2} = \frac{s_1}{\frac{1}{12}d_1^3 + 0.2\left(\frac{d_1}{2} - 0.17\right)^2} \qquad (14)$$

l	d	l	d	l	d
1	4.10	10	1.72	19	1.20
2	3.72	11	1.60	20	1.16
3	3.38	12	1.50	21	1.13
4	3.05	13	1.42	22	1.10
5	2.78	14	1.37	23	1.07
6	2.51	15	1.33	24	1.05
7	2.28	16	1.31	25	1.03
8	2.07	17	1.29	27	1.01
9	1.90	18	1.23	29	1.00

The calculations that follow can be effected very expeditiously and accurately by aid of Barlow's "Table of Squares" and Crelle's "Rechentafeln," which last

gives the product of three figure numbers by similar numbers. A calculating machine can of course replace the last named book.

15. It would often save time in the end if the values of

$$\tfrac{1}{12} d^2 + 0.2\left(\frac{d}{2} - 0.17\right)^2$$

for various values of d as 1.0, 1.2, 1.4, 1.6, 1.8, 2.0, 2.2, 2.4. be computed at once and laid off as ordinates on cross-section paper, the values of d being laid off as abscissas. It is advisable, too, to find by (14), at the first, by trial, such a value of s_1 that s at the crown will not become so small as to lead to a great number of divisions. Thus if s_1 is assumed $= 7$ feet, s at the crown will be found to be only 0.184; therefore $s_1 = 14$ was next tried, and the work then proceeds as follows :

For $s_1 = 14$, d_1 is estimated at $l = 7$, $\therefore$ $d_1 = 2.28$ (from table), $\therefore$ the right member of (14) reduces to 11.9, and the

relation to be satisfied throughout the arch ring is,

$$s = 11.9\left[\tfrac{1}{12}d^3 + 0.2\left(\frac{d}{2} - 0.17\right)^2\right] \quad (15)$$

At the crown $d = 1$, $\therefore s = 1.25$, which may lead to a satisfactory division; hence we proceed, taking up the division s_2 next s_1.

Try $s_2 = 3$, $\therefore d$ at the middle of s_2, where $l = 14 + 1.5 = 15.5$ from the springing, is found from the table (by interpolation) to equal 1.32, $\therefore$ by (15), $s = 2.86$. For $s = 2.86$, there is no change in d, $\therefore s_2 = 2.86$, so that it is $(14 + 2.86)$ feet to the beginning of division s_3. Assume $s_3 = 2.3$, $\therefore d$ at $l = 16.86 + \frac{1}{2}(2.3) = 18.01$, is 1.23, $\therefore$ by (15), $s = 2.32$, $\therefore$ call $s_3 = 2.32$. Proceed thus and find successively:

$s_1 = 14$, $s_2 = 2.86$, $s_3 = 2.32$,
$s_4 = 1.95$, $s_5 = 1.67$, $s_6 = 1.49$,
$s_7 = 1.40$, $s_8 = 1.19$, $s_9 = 1.26$,
$s_{10} = 1.25$.

The sum of these lengths is 29.52, or 0.52 in excess, as it is exactly 29 feet from crown to spring measured along the centre line.

A closer approximation can now be made by assuming s_1 a little less than 14. Take $s_1 = 13.85$, $\therefore$ $d_1 = 2.30$ at the middle of s_1 or at $l = \frac{1}{2}(13.85) = 6.92$, and the right member of (14) reduces to 11.5. Therefore (15) is now replaced by

$$s = 11.5\left[\tfrac{1}{12}d^2 + 0.2\left(\frac{d}{2} - 0.17\right)^2\right]$$

s	l at end of s	l at middle of s	Corresponding d
$s_1 = 13.85$	13.85	6.92	2.30
$s_2 = 2.81$	16.66	15.25	1.33
$s_3 = 2.29$	18.95	17.80	1.24
$s_4 = 1.89$	20.84	19.90	1.16
$s_5 = 1.64$	22.48	21.66	1.11
$s_6 = 1.47$	23.95	23.21	1.07
$s_7 = 1.35$	25.30	24.62	1.04
$s_8 = 1.31$	26.61	25.96	1.03
$s_9 = 1.24$	27.85	27.23	1.01
$s_{10} = 1.21$	29.06	28.45	1.00

We proceed as before in finding s_2, s_3, . . . s_{10}. The results are conveniently written out as in the adjoining Table.

Here, "l at end of s" is found by summing s. To find l at middle of s_4 (say), add one half of s_4 or $\frac{1}{2}(1.89) = 0.95$ to l at end of s_3, or 18.95, giving $18.95 + .95 = 19.90$, at which point d is found from Table to equal 1.16. With this value of d, s_4 is computed from the last formula (by aid of the diagram mentioned, if drawn), and if the result is not as assumed, try again until the assumed and computed values of s_4 agree. Similarly for the other divisions. The work is very brief on a second trial as the previous values of s_2, s_3, . . . , afford some guide in choosing the new ones.

A third trial is not at all necessary here, as the excess 0.06 foot of $s_1 + s_2 + \ldots + s_{10} = 29.06$, over the actual length 29.00, can be distributed amongst the various divisions sufficiently accurately by comparing the corresponding values of s

in the first and second trials. The values of s finally selected are,

$s_1 = 13.83$, $s_2 = 2.80$, $s_3 = 2.28$,
$s_4 = 1.88$, $s_5 = 1.64$, $s_6 = 1.46$,
$s_7 = 1.35$, $s_8 = 1.31$, $s_9 = 1.24$,
$s_{10} = 1.21$.

These lengths are laid off in succession from the centre of each springing joint toward the crown along the centre line of the arch ring.

The number of divisions found for the half arch (ten) is a minimum for accurately locating the pressure line for vertical loads. Fourteen to twenty should be used for temperature stresses. In fact, for temperature stresses, it will not add greatly to the labor to take from twenty to thirty divisions as we shall see further on.

16. It need not cause surprise that s_1 is so large compared with s_{10}, when it is observed that the moment of inertia at the spring is 81 times that at the crown. The resistance to bending is thus

so much greater near the abutment that we should expect the elastic curve and the pressure line to be determined mainly by the upper half of the arc.

If the abutment should be included as part of the arch, its influence on the pressure line would be found to be almost nil, which is in fact the reason why it is treated as inelastic and immovable.

As the radial depths of arch ring approach equality everywhere, s_1, s_2, . . . , approach equality, and for a constant depth they are exactly equal.

17. For any kind of a *solid, homogeneous* arch of any material, stone, concrete, etc., the trial and error method used for dividing up the arch ring is similar to that above, only as $I_2 = o$, (14) is replaced by,

$$\frac{s}{d^3} = \frac{s_1}{d_1^{\ 3}} \qquad (16),$$

the work is much simpler, as d^3 can at once be taken out a table of cubes, and a diagram is unnecessary.

Example.—Divide the arch ring above, omitting the steel ribs, to satisfy (16). Let the number of divisions of the half-arch be at least ten and not to exceed sixteen. The Table of Art. 14 gives the values of *d*.

COMPLETE GRAPHICAL TREATMENT FOR A STEEL CONCRETE ARCH WITH PARTIAL LOAD.

18. The arch ring examined in Articles 14 and 15, is shown in Fig. 6 (Plate). The intrados has three centres. The central portion subtends an angle of $30°20.4'$ with a radius of 47.77 feet; the other two radii are 23.65 feet each. The span is 50 feet, rise 10 feet, and the radial depths of arch ring are as given in Art. 14, the depth at the crown being 1 foot, at the ends of the central portion 1.33 feet, and at the springs 4.5 feet, circular curves corresponding, forming the extrados. The steel ribs are placed as shown in Art. 14.

Let us test the strength and stability of this arch when the backing extends 1

foot above the crown everywhere, for a live load of 140 pounds per square foot of roadway extending from the left abutment to the crown. The ordinates from the extrados to the upper limit of the backing should be reduced in the ratio of the specific gravity of the backing to that of the arch ring, so that the new upper limit corresponds to a backing whose density is the same as that of the arch ring. The partial loads are then computed as below. In this example, this reduction was not made, so that the backing was assumed of the same density as the concrete, 140 pounds per cubic foot.

19. The lengths s_1, s_2, . . . (Art. 15), having been laid off in succession along the centre line of the arch ring from either springing to the crown, find the centres of each of the divisions s_1, s_2, . . . , and mark them in order from the right springing to the left, a_1, a_2, . . . a_{20} as shown on the Figure.

Draw vertical lines through a_1, a_2 . . a_{20}, and compute the areas in square feet in-

cluded between consecutive verticals, the intrados, and the upper limit of the (reduced) backing. Multiply these weights by 140 to find the loads P_1, P_2, . . . P_{10} to right of crown, and lay off midway between the points a_1, a_2, . . . a_{10} ; the last one coming midway between a_{10} and the crown. These loads correspond to a slice of the arch 1 foot thick perpendicular to the plane of the paper.

When a_1 and a_2 are as far apart as in the figure, the vertical through the centre of gravity of this large trapezoid, should be found by the usual graphical construction, and P_1 laid off along it. The same remark applies to the remaining areas nearer the abutments ; or the position of the load corresponding can be found by taking moments of the partial areas about any convenient vertical. The loads P_{11} . . . P_{20} to left of the crown are equal respectively to P_{10} . . . P_1 augmented by 140 times the width of the corresponding division.

From Art. 12, the moments $M = H.t$

must be found accurately at the middle of s_1, s_2, . . . , or at a_1, a_2, . . . a_{19},a_{20}, which can be done by the division of the arch just explained as is apparent from the equilibrium polygon to be presently drawn.

A usual division of the arch requires t to be measured *at* the load, which is a most unfortunate selection. If the true equilibrinm *curve* for infinitesimal divisions be supposed drawn, it will always fall below the equilibrium polygon corresponding to P_1, P_2,... , except at a_1, a_2 ... , (which are points on it), and the greatest departure is at a load.

Such a division of the arch leads to values of t as far removed from the true ones as possible, whereas the method adopted here gives no inaccuracy from this cause.*

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* See the author's "Theory of Voussoir Arches," 2nd ed., for the most accurate scheme for arches not so flat as shown in Fig. 6, and for sections not vertical as taken here for simplicity. The theory which determines the subsequent constructions is given in full on pages 154 to 174. The author's "Solid and Braced Elastic Arches" may also be referred to.

20. The successive loads P_1, P_2, . . . P_{20} are now laid off on a vertical to the left of the arch to scale of loads. For convenience, assume the pole distance some whole number, 10,000 lbs. in this case, and lay off this horizontal thrust from the lower end of P_{10} on the load line, to the right to fix the "trial pole." This assumes a horizontal thrust at the crown. The equilibrium polygon b_1, b_2, . . . b_{10}, . . . b_{20}, can be drawn in the usual manner, but it is most accurately laid off by computing the sum of the moments of the loads P in foot pounds, acting from the crown up to a certain load P_s about P_s, dividing by trial $H = 10{,}000_2$ and laying off the ordinate (to scale of distance) from a horizontal through b_{10}, b_{11}, along the line of action of P_s to fix one point. Similarly for the others. Let P_r be the load next P_s at a distance a from it, P_r lying nearer the crown than P_s. Call the sum of the loads from the crown up to and including $P_r = R$, and suppose their resultant acts c feet from P_r. Then the moment of all

the loads from the crown up to P_s about $P_s = M_s = R(c + a) = Rc + Ra = M_r + Ra$, where M_r is the moment of all loads from the crown up to P_r about P_r. This formula, $M_s = M_r + Ra$, enables the moments to be quickly computed by working from the crown in either direction.

Thus,

$$M_{12} = P_{11} \times 0.95 = 252.7$$

$$M_{13} = M_{12} + (P_{11} + P_{12})\, 1.27 = 1247.1$$

&c. &c.

The equilibrium polygon laid off by computation, can now be tested by the usual construction and verticals through $a_1, a_2 \ldots$, drawn to intersections $b_1, b_2, \ldots$, with it.

21. The reason for the following procedures will be given further on (Art. 30). Draw a straight line from $b_1 = v_1$ to $b_{20} = v_{20}$, and find $R = (v_2b_2 + v_3b_3 + \ldots + v_{19}b_{19})$ or the sum of the ordinates from $v_1\, v_{20}$ to $b_2, b_3, \ldots b_{19}$ to be 169 to the scale of distance. Next regard v_2b_2, $v_3b_3, \ldots v_{19}b_{19}$ as forces, and find the

position of their resultant R by taking moments about the vertical AB through the crown. This is conveniently done by measuring the horizontal distances z_2, z_3, . . ., from AB to b_2, b_3, . . . (always measure distances to hundredths of a foot) and dividing $[(v_{19}b_{19} - v_2b_2)\,z_2 + (v_{18}b_{18} - v_3b_3)\,z_3 + \ldots + (v_{11}b_{11} - v_{10}b_{10})z_{10}] =$ 38.0797 by R = 169 to get the distance (= 0.225 feet) the resultant acts to left of AB.

Differences, as $(v_{19}b_{19} - v_2b_2)$, can be conveniently found with dividers, or by marking off v_2b_2 on a strip of paper and laying off along $v_{19}b_{19}$.

22. The object now is to find a closing line mm_1, such that if the ordinates from v_1v_{20} to m_1m, through the points b_1, b_2, . . . b_{20}, are treated as forces, their resultant will exactly equal R in magnitude and coincide with it in position. To this end a trial "closing line" nn_1 is first drawn, and likewise a straight line from n to v_1, dividing the ordinates into two sets, viz.; those found in the triangle $v_{20}nv_1$, whose

resultant is called "trial T," and those included in the triangle nv_1n_1, whose resultant is designated "trial T_1". Find the resultant of the ordinates (treated as forces) from the line v_1v_{29} to the line v_1n at the points v_2, v_3, . . . v_{28} in amount, by marking off in succession, along the edge of a sheet of paper, and measuring the sum to the scale of distance used in laying off the arch. This scale is to be used throughout in measuring ordinates and abscissas, whether treated as forces or otherwise. In this way find "trial T" to be 93.00. By taking moments about AB as above, it is found that trial T acts 4.57 feet to left of AB or 4.35 feet to left of R, as marked on the drawing. Differences between the ordinates at same distance from AB, as, e.g., those at v_{12} and v_9 are conveniently marked off by drawing a straight line from D (where nv_1 meets AB) to v_{20}.

Similarly find trial T_1 by adding the ordinates through b_1, b_2, . . . b_{19} included between the lines nv_1 and nn_1 to be 58.9.

It evidently acts as far to the right of AB as trial T does to the left, or 4.57 feet. This is so because corresponding ordinates are at the same distance from AB, and the *position* of trial T_1 is not affected by shifting n_1 so as to make $v_1n_1 = v_{20}n$, since all ordinates are increased in the same ratio, and the quotient of sum of moments divided by sum of forces remains the same. By similar reasoning, if n is shifted to its true position m, the sum of the ordinates from v_1v_{20} to v_1n = "true T;" and (the ordinates being treated as forces) in position, the resultant acts 4.35 feet to left of AB, the same as for trial T.

The position and amount of trial T_1 is not changed by shifting n to m. Its position is not changed on afterwards shifting n_1 to m_1; its true position and true T_1 = sum of ordinates from mv_1 to mm_1.

By the assumption at the beginning of this article, it follows that R must be the resultant in position and magnitude of T

and T_1; ∴ taking moments about T_1 and T in turn,

$$\text{true T} = \frac{169 \times 4.79}{9.14} = 88.57;$$

$$\text{true T}_1 = \frac{169 \times 4.35}{9.14} = 80.43.$$

Hence lay off,

$$v_{20}m = \frac{\text{true T}}{\text{trial T}} v_{20}n = \frac{88.57}{93} v_{20}n,$$

and,

$$v_1m_1 = \frac{\text{true T}_1}{\text{trial T}_1} v_1n_1 = \frac{80.43}{58.9} v_1n_1,$$

and it will follow that the ordinates of the triangles will all be changed in the ratio of true T to trial T for triangle $v_1v_{20}n$, and of true T_1 to trial T_1 for triangle v_1nn_1 or v_1mn_1. Therefore, $R = \Sigma(vb)$ is now the resultant in position and magnitude of all the ordinates from v_1v_{20} to mm_1 or ordinates of the type vm. The closing line mm_1 is thus finally located.

23. Take O, the centre of the left

springing joint, as the origin of co-ordinates, x horizontal, y vertical, and let x_1, x_2, . . . x_{20} be the horizontal distances from O to the verticals through a_1, a_2, . . . a_{20}, or generally, let x be the abscissa of a; then since the construction of Art. 22 gives:

$$\Sigma\,(vb) = \Sigma(vm), \quad \Sigma(vb.x) = \Sigma(vm.x),$$

therefore

$$\Sigma\,(vb - vm) = 0, \quad \Sigma\,(vb - vm)x = 0\ ;$$

or the conditions,

$$\Sigma\,(mb) = 0, \quad \Sigma\,(mb.x) = 0, \qquad (17)$$

are fulfilled.

Here, $mb = vb - vm$, $\therefore$ ordinates mb measured above mm_1 must be treated as positive, those below negative.

As a check on the position of mm_1, see if the condition $\Sigma(mb) = 0$ is fulfilled, or if the sum of the mb's above mm_1 is equal to the corresponding sum below. The sum is very quickly made by marking off successive mb's along the straight edge of a sheet of paper.

24. If a symmetrical arch is loaded with its own weight only or, in addition, with a uniform live load, the equilibrium polygon b is symmetrical with respect to AB; hence R acts along AB, and mm_1 is parallel to v_1v_{20}.

Here, $\Sigma vb = \Sigma vm = 20vm$; $\therefore$ divide the sum of the ordinates of type vb to one side of the crown by 10 (in this example) to find $v_1m_1 = v_{20}m$.

As above, $\Sigma mb = 0$, $\Sigma(mb.x) = 0$.

25. Let now a line kk_1 be located in a similar manner with respect to the points $a_1, a_2, \ldots a_{20}$ on the neutral line of the arch ring.

This is best done by measuring the ordinates $y_1, y_2, \ldots y_{10}$ from OO_1 (the line connecting the centres of the springing sections) to $a_1, a_2, \ldots a_{10}$ and dividing their sum by the number of ordinates, (10), giving the distance e above OO_1 to kk_1.

In this example,

$$e = \frac{\Sigma_0^{\frac{l}{2}} y}{10} = 8.14.$$

since $\quad \Sigma_0^{\frac{l}{2}} = 10e, \quad \Sigma_0^{\frac{l}{2}}(y - e) = 0,$

$$\therefore \quad \Sigma_0^{\frac{l}{2}}(ka) = 0,$$

and from symmetry, $\quad \Sigma_0^{l}(ka.x) = 0.$

Hence the conditions,

$$\Sigma_0^{l}(ka) = 0, \quad \Sigma_0^{l}(ka.x) = 0 \qquad (18)$$

for the line kk_1, as thus located, are fulfilled, ordinates ka, referring to ordinates from kk_1 to a_1, a_2, . . . ; ordinates above kk_1 being regarded as plus, those below minus.

26. The next step is to find $\Sigma(ka.y)$ for the entire arch. This may be written, $\Sigma(y - e)y = \Sigma y^2 - e\Sigma y = 2(684.79 - 8.143 \times 81.43) = 41.90$; the sum being taken for one half the arch and doubled.

27. A similar sum, $\Sigma(mb.y)$ must now be made out for the entire arch ; mb representing an ordinate from mm_1 to poly-

gon b, to be considered plus when measured above mm_1, minus below.

$\therefore \quad \Sigma(mb.y) = (mb_{10} + mb_{11})y_{10} + (mb_9 + mb_{12})y_9 + (mb_8 + mb_{10})y_8 + (mb_7 + mb_{14})y_7 + (mb_6 + mb_{15})y_6 + (mb_5 + mb_{16})y_5 + (mb_4 + mb_{17})y_4 - (mb_3 + mb_{18})y_3 - (mb_2 + mb_{19})y_2 - (mb_1 + mb_{20})y_1$. On measuring ordinates to the scale of distance and computing the sum of the products above, we find $\Sigma(mb.y) = 90.90$.

28. It is a principle of the equilibrium polygon that if the ordinates mb are altered in a given ratio, the pole distance is altered in the inverse ratio. The ordinates mb are now altered in the ratio 41.9 to 90.9, which is easily done graphically) and the pole distance in the ratio 90.9 to 41.9.

The new ordinates mb are laid off vertically above or below kk_1, according to sign, to find all the points $c_1, c_2, \ldots c_{20}$ in the same vertical with $a_1\ a_2, \ldots a_{20}$ respectively. The points c are points in the true equilibrium polygon for the arch, for the loads assumed. The centre of

pressure on the crown joint is similarly found.

To find the true pole, draw from the trial pole, Fig. 6, a parallel to mm_1 to intersection with the load line, then horizontally to the right, a line of length $= 10000 \times \frac{90.9}{41.9} = 21695$ lbs., measured to the scale of loads. The new pole is thus located and the true horizontal thrust of the arch found to be

$$H = 21695 \text{ pounds.}$$

Beginning at the centre of pressure at the crown, the equilibrium polygon $c_1, c_2, \ldots c_{20}$ can be tested by the usual graphical construction.

Those familiar with Prof. H. T. Eddy's "Researches in Graphical Statics," will recognize my indebtedness to him in the general treatment of the equilibrium polygon *b* and its final location in true position on the arch.

ARCH LOADED WITH ITS OWN WEIGHT.

29. The arch loaded with its own weight only can be easily treated when the previous constructions have been made, since the part of the equilibrium polygon required, $b_1, \ldots, b_{10}$ has already been drawn. The closing line mm_1 is quickly determined, as in Art. 24, and the line kk_1 is already established (see Art. 25). Then as in Arts. 26 and 27, find $\sum(ka.y)$ and $\sum(mb.y)$ for the half arch only, and proceed as in Arts. 28 and 29 to alter all the mb's in the ratio $\sum(ka.y)$ to $\sum(mb.y)$, and lay off vertically above and below kk_1 to fix the points $c_1, \ldots, c_{10}$, and change the pole distance in the inverse ratio to find the true pole (Art. 29). In this example, the closing line mm_1 is found to be $72.22 \div 10 = 7.22$ above v_1, $\therefore \sum_0^{\frac{l}{2}}(mb.y) = +38.05$ and (Art. 26), $\sum_0^{\frac{l}{2}}(ka.y) = +20.95$. Hence the mb's are all altered in the ratio 38.05 : 20.95 and the true horizontal thrust (pole distance) $= 10{,}000 \times \dfrac{38.05}{20.95} = 18{,}160$ pounds. On laying off the new mb's from kk_1, it is found that

$a_1c_1 = +0.1$, $a_2c_2 = +0.1$, $a_3c_3 = +0.07$, $a_4c_4 = +0.06$, $a_5c_5 = +0.03$, $a_6c_6 = a_7c_7 = 0$, $a_8c_8 = -0.09$, $a_9c_9 = -0.1$, $a_{10}c_{10} = -0.1$ distances above a being plus; below, minus.

30. Demonstration. Since the equilibrium polygon b with ordinates all altered in the same ratio is now in position on the arch as polygon c, the closing line mm_1 now coinciding with kk_1, we have from Eq. (17), Art. 23,

$$\Sigma(kc) = 0, \quad \Sigma(kc.x) = 0.$$

Also from Art. 25, Eq. 18, the line kk_1 was located to satisfy the conditions,

$$\Sigma(ka) = 0, \quad \Sigma(ka.x) = 0,$$

the summation extending over the entire arch. On subtracting the last equations from the preceding,

$$\Sigma(ac) = 0, \quad \Sigma(ac.x) = 0,$$

and it is seen that the first two conditions, Art. 14, Eq. (13), for an arch without hinges, are fulfilled. Finally, since each mb has been altered in the ratio 41.9 to 90.9 to the corresponding kc, $\Sigma(mb.y) =$ 90.9 has been changed to $\Sigma(kc.y) = 41.9$.

But, Art. 26, $\Sigma(ka.y) = 41.9$; therefore, by subtraction,

$$\Sigma(ac.y) = 0,$$

and the third condition of Art. 14, Eq. 13, is fulfilled by equilibrium polygon c. It is thus the true one for the arch "fixed at the ends," and with no hinges.

Unit Stresses, etc.

31. The normal component T at any radial section, as the one through a_1, is found by resolving the thrust there, as given by the proper ray of the force diagram, measured to the scale of force, into two components normal and parallel to the section, and measuring the normal component to scale. Thus T at $a_1 =$ 24470 lbs. The moment, $M = Ht =$ $H.a_1c_1$ at a_1, is found by measuring a_1c_1 to scale of distance, giving $a_1c_1 = .41$, and multiplying by $H = 21700$ to get the moment $M = 8897$ in foot pounds. The stress on the concrete at the extrados or

intrados is now found from Art. 11, Eq. (9), to be, as $d_1 = 2.3$

$$= \frac{24470}{2.3+0.2} \pm \frac{8897 \times 1.15}{\frac{1}{12}(2.3)^3 + 2(0.98)^2}.$$

$$= \left.\begin{matrix} 18,274 \\ 1,306 \end{matrix}\right\} \text{ pounds per square foot,}$$

or 127 lbs. per sq. in. compression at the extrados and 10 lbs. per sq. in. compression at the intrados.

This will suffice to explain the general method of procedure at any section.

Note that ac is now regarded as + when c is above a, — otherwise.

At the crown, $ac = -0.04$, and at a_8, $a_8c_8 = -0.13$, giving compression at the intrados on the concrete = 154 at the crown and 214 at a_8, and likewise compression at the extrados, 97 at the crown and 34 at a_8, all in pounds per square inch.

The section at a_8 was found to give the maximum compression at the intrados. These stresses will ultimately be combined with the stresses due to temperature.

32. It may be thought that a good check should be afforded by actually measuring a_1c_1, a_2c_2, . . . $a_{20}c_{20}$ accurately to scale and computing $\Sigma(ac)$, $\Sigma(ac.x)$, and $\Sigma(ac.y)$, which should each approximate zero if the points c have been correctly located. As it is perhaps impossible to locate any c to two or three hundredths of a foot, the check is not a good one as the distances ac are very small at most points. It will be found in this case, however, that although $\Sigma(ac)$, $\Sigma(ac.x)$, $\Sigma(ac.y)$ are not zero, yet if each ac is diminished by 0.02 foot, the sums change sign, so that it seems here that the points c have been located within two hundredths of a foot of their correct positions.

33. The resultant on the right springing section is found by producing the resultant at c_1 to intersection with the vertical through the centre of gravity of the remaining part of the arch and load to the right of the vertical through a_1, combining the above resultant with the

weight of this remaining portion (this is effected on the force diagram) and drawing a parallel to this final resultant through the intersection mentioned to the springing. Similarly we proceed for the left springing. The centres of pressure at both springs are well within the middle third, so that the radial depth there could be diminished to about 4 feet. The greatest departure of the polygon c from the centre line a is at a_1 already noted. Here c_1 is above a_1. Polygon c then passes through a_2 and falls below the neutral line, its maximum departure being $a_8c_8 = -0.13$. At the crown it is 0.04 below, at a_{12} it crosses the neutral line and remains above until near the left springing, the maximum $+t$ being $a_{17}c_{17} = 0.13$.

34. The line of the centres of pressure varies for different arches, but for the upper half of the arch, its position relative to the neutral line is about the same for all circular arches similarly loaded.

For a three-centred arch of 89.55′ span and 15.10′ rise, having a radial depth at the crown 1.83′, at 25 feet horizontally from the crown 2.29′ and at the springs 7.56′; and loaded with 125 pounds per foot on the left half, the pressure line was very similar to the one above except that at the left springing, the centre of pressure was at the lower middle third limit, at the right springing it was slightly below the centre and it passed exactly through a_1.

The backing here extended 1 foot above the crown and its density was taken as 0.8 that of the concrete. The central angle was 32°06.2′, corresponding to the radius 90.41, the remaining portion of the intrados being described with a radius 46.55. Also the steel bars were placed as above,

$$A_2 = \frac{1}{72} \text{ sq. ft. and } 20\ I_2 = \frac{10}{36}\left(\frac{d}{2} - 0.17\right)^2.$$

The stresses on the concrete in pounds per square inch at the left springing were 129 at the intrados and 4 at the extrados, both compression; at the crown the stresses were 288 compression at the intrados and 59 at the extrados. There was no tension exerted anywhere.

35. Arches of variable section of any material, as concrete, stone, brick or

steel, without the ribs enclosed, are to be treated exactly as above, except that $A_2 = 0$, $I_2 = 0$ in the formulas. In Art. 17 the method of dividing up the arch ring is indicated.

The brick arch should be built of the best brick in Portland cement, with thin joints ($\frac{1}{4}$ inch say). An approximation is of course introduced in regarding the brick arch as a homogeneous one, but the Austrian experiments go to show that the theory applied on this hypothesis is sufficiently reliable. In the brick or stone arch, if the mortar cannot take appreciable tension, the theory is only valid when the centres of pressure on the joints are found to lie everywhere within the middle third of the arch ring. The stone arch of constant section is fully treated in the author's "Theory of Voussoir Arches" as regards vertical loads.

In all arches, the theory considers the backing to act vertically only. This is nearly true if the weight of roadway with its load is carried to the arch through

pillars of steel, stone, etc. In the case of earth backing, a certain passive resistance to the tendency to spread at the haunches is exerted by the earth which may be taken as adding to the stability. This becomes very pronounced when a series of longitudinal spandrel walls form part of the backing. This may be considered as adding so much to the factor of safety, since it cannot be directly evaluated.

Skew arches are treated exactly as right arches, the span being taken on the skew and not at right angles to the axis of the arch. Full centre or elliptical arches are always built with solid masonry backing up to a certain height. The part above this can be treated as the arch proper and the part below as the abutment. Attention has been called to the reason for this in Article 16.

TEMPERATURE STRESSES.

36. The stresses due to temperature in in the arch with no hinges are very high and should be carefully considered. The

theory is simple. Suppose the arch without weight, and that it exactly fits between the skewbacks without stress anywhere, at a certain *mean* temperature. That is, if the arch was laid on its side on a horizontal platform, with the assumed span, skewbacks and rise, it would be without stress at the mean temperature. Denote the greatest deviation above or below this by t° in Celsius degrees, and the expansion of the material of the arch for a unit of length and one degree by ε. The total change in length of span of neutral line is $l\varepsilon t^{\circ}$, where l represents the length of span of neutral line in feet, since this is made up of the horizontal projections of the changes for each elementary portion of the arch.

As the abutments resist this tendency to a change of shape, a horizontal thrust and bending moment will be experienced at each abutment. For a symmetrical arch, these will be the same for each abutment, and the result will not be altered if we conceive horizontal forces H,

to act at a distance e above the centre of each springing section, both acting inwards for a rise, and outwards for a fall of temperature. On conceiving at the centre of each section at the springs, two horizontal forces H opposed to each other, the force H acting a distance e above this centre with one of the opposed H's, forms a couple whose bending moment is He, and the remaining H gives the horizontal thrust at the abutment. If the section is unsymmetrical, substitute throughout for centre of the section at the springs, the point where the neutral line meets the section. If the arch ring is divided into lengths s_1, s_2, . . ., as in Art. 14, so that $s \div (\mathrm{I}_1 + n\mathrm{I}_2)$ is con-tant, then as before, since the end tangents are fixed and the vertical displacement of the end sections zero,

$$\therefore \quad \Sigma_0^l t + = 0, \quad \Sigma_0^l (tx) = 0.$$

In Fig. 6 (plate), where this division of the arch ring was effected, if we suppose the H at either end to act along kk_1,

the two conditions are fulfilled, since

$$M = H.ka \quad \therefore \quad t = ka,$$

and by Art. 25, kk_1 was located so that

$$\Sigma_0^l(ka) = 0, \quad \Sigma_0^l(ka.x) = 0.$$

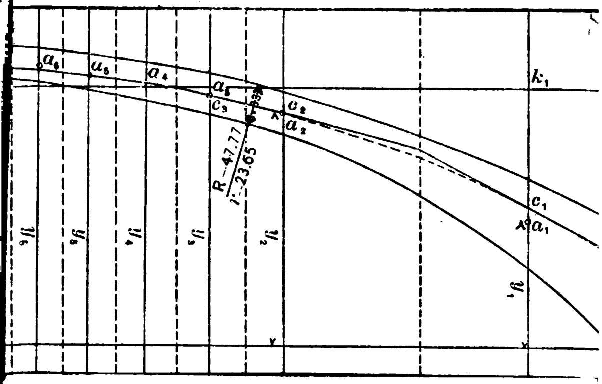
A rise of temperature increases the span for a free arch; hence between rigid abutments, the new span must now be decreased by the same amount to fit it in place. Similarly for a fall of temperature. In this example, the distance between centres of sections at the springs $= l = 53.6$ feet.

By Art. 12, the change of span of neutral line $l\varepsilon t^\circ$ is given by,

$$l\varepsilon t^\circ = 2\Sigma \frac{Mys}{E_1(I_1 + nI_2)} =$$

$$2H \frac{s}{E_1(I_1 + nI_2)} \Sigma(ka.y),$$

the summation extending over the half span.



a_6
a_5
a_4
a_3
c_3
c_2
a_2
k_1
c_1
a_1
R = 47.77
r = 23.65
y_6
y_5
y_4
y_3
y_2
y_1

Since $ka = y - e$,

$$\Sigma(ka.y) = \Sigma y^2 - e\Sigma y,$$

$$\therefore \mathrm{H} = \frac{\mathrm{E}_1 l\varepsilon t^{\circ}}{2(\Sigma y^2 - e\Sigma y)} \frac{\mathrm{I}_1 + n\mathrm{I}_2}{s} \qquad (19)$$

(See Art 25 for e).

From Art. 26,$2(\Sigma y^2 - e\Sigma y) = 41.90$ for the arch of Fig. 6. Let $e = 0.000012$, and suppose a fall of 35° C. (corresponding to 63° F.), ∴

$l\varepsilon t^{\circ} = 53.6 \times 0.000012 \times 35 = 0.022512$.

The modulus of elasticity of the concrete will be taken at 1,400,000 × 144 lbs. per square foot.

By Art. 15, $s \div (\mathrm{I}_1 + n\mathrm{I}_2) = 11.5$. Substituting the above values in the formula, and we find, H = 9418.6 pounds, for the pull exerted for an arch 1 foot thick for a fall of temperature of 63° F. Let us assume a mean temperature of 53° F, and the extremes, 93° F. and − 10° F., corresponding to the fall 63° F. and a rise 40° F. Then H for 40° F. rise is

$9420 \times \frac{40}{63} = 5980$, and the correspond ing stresses are in the ratio 40 : 63 to those corresponding to 63° fall, but of opposite character.

In finding the stresses by aid of Eq. (9), Art. 11, the work proceeds as in Art. 31, H being resolved into components $\perp$ and $\parallel$ to the section. The normal component of H is T of the formula, and $M = H.ka$.

It conduces to clearness, to find the character of the stresses at a section of the right half of the arch, as that at a_1, to conceive the left half removed, and to replace its action on the right half by a single force H along kk_1, acting to the left for a fall and to the right for a rise of temperature. As kk_1 lies above a_1, this causes compression at the intrados, tension at the extrados for a fall, and the reverse for a rise of temperature.

On computing all the stresses due to temperature and combining results with those due to the loads, it is found that

the maximum tension ever experienced is at the intrados at the crown, corresponding to a 63° F. fall of temperature. Its amount is 213 pounds per square inch. The maximum compression corresponds to 40° rise of temperature and is at the intrados for section at a_8, the stress being 428 pounds per square inch. The stresses where no live load is on the bridge, should be ascertained and combined with the temperature stresses as above. The rise or fall of temperature to be allowed will be discussed in Art. 40.

37. To ascertain the approximation involved in the above method for finding H for temperature stresses, a solid homogeneous circular arch of constant section was assumed, the neutral line for the half arch divided into 16 equal parts and, for the dimensions assumed and a rise of temperature of 30° C., the thrust was found to be $H = 201.6$ tons. By the formula below it was 204.6 tons, a very satisfactory agreement.

The formula is exact, except that the

uniform compression due to T is neglected as in the method above, and is given by Greene, Howe, Winkler, and others. (Winkler's approximate formula gives results differing widely from this exact one.) The formula involves the quantities used above, also the radius r of the neutral line and the half central angle α, and in the most convenient form is,

$$H = \frac{2\ EIt^0\varepsilon}{r^2\left(\frac{\alpha}{\sin\ \alpha} + \cos\ \alpha - 2\frac{\sin\ \alpha}{\alpha}\right)}$$

A six or seven place table must be used in computing the parenthesis in the denominator.

38. For the arch of variable section having any curves for intrados and extrados, the formula of Art. 36 can be put into another form, which can be used when $s \div (I_1 + nI_2)$ is not constant. Let $I = (I_1 + nI_2)$ for brevity, and putting $s \div I$ under the summation signs, Eq. (19) can be written,

$$H = \frac{E_1 l\varepsilon t^{\circ}}{2\left[\Sigma\left(y^2\,\frac{s}{I}\right) - e\Sigma\left(y\,\frac{s}{I}\right)\right]} \qquad (20)$$

Formula (10), Art. 12, gives

$$\Sigma\,\frac{H.ka.s}{E_1 I} = 0, \quad \therefore\ \Sigma\left(ka\,\frac{s}{I}\right) = 0,$$

$$\therefore\ \Sigma(y-e)\frac{s}{I} = 0, \quad \Sigma y\frac{s}{I} = e\Sigma\frac{s}{I}$$

$$\therefore\quad e = \frac{\Sigma y\,\frac{s}{I}}{\Sigma\,\frac{s}{I}}.$$

The summation in the formulas of this article, extends over half the span only on account of symmetry.

[This formula for e reduces to the previous value, Art. 25, when $s \div I$ is constant, as it becomes then $e = (\Sigma y) \div n$, where Σy is the sum of the ordinates on the half arch and n is their number.]

On substituting the above value of e in (20), we are conducted to the formula

given by Howe, page 49, in his valnable work on Arches.

In using this formula (20), it is most convenient to divide the neutral line into parts of equal length, scale off the ordinates to the centre of each division above the span line of the neutral axis (generally from centre to centre of skewbacks), and tabulate quantities as follows :

Point	Dist. from abutment	I	$\frac{1}{I}$	y	y^2	$\frac{y}{I}$	$\frac{y^2}{I}$

Formula (20) saves the tentative division of the arch ring, to make $s \div I$ constant. It furnishes some check upon the former method. To compare them, a steel-concrete arch of 100 ft. span, 3 ft. depth of key, and 7.5 ft. radial depth at spring was treated by both methods, using 14 divisions of the semi-arch by either method. The first one ($s \div I$ constant), gave a result about 4 per cent. smaller

than the second where *s* was taken constant.

These determinations should give confidence in the methods used. It was found however that it was highly desirable to use as many divisions as possible, say 20 to 30 for the half arch, to attain accuracy by use of (19). Possibly 16 should be a minimum and then the result will be too small by a few per cent.

PROPERTIES OF CONCRETE.

39. Trautwine gives the following crushing loads for concrete, reduced to pounds per square inch :

Age in months	1	3	6	9	12
Pounds per sq. in.	230	623	1010	1320	1560

Under favorable conditions, these figures may be increased 50 to 100 per cent. Several times these figures have been found by some experimenters. Possibly 500 pounds in two months can safely be counted on in practice.

The experiments on tensile strength

are very meagre. About 300 pounds per square inch ultimate may be given as an average, and 150 to 200 is recommended by some as a safe stress, and 50 to 60 by others. The modulus of elasticity has been given all the way from 1,400,000 to 4,000,000 pounds per square inch for concrete. The smaller figure refers to experiments on a 74.5 feet span and is possibly better adapted to practice. The modulus for permanent set ranged from 9,000,000 to 32,500,000. Mr. Hyatt proved the expansion of steel and concrete to be equal, whether under load or fire. Bauschinger however gives the coefficient of expansion for 1° C., referring to 1 : $2\frac{1}{2}$: 5 concrete, as $\alpha = 0.0000088$, whereas a usual figure for iron is $\alpha = 0.000012$. The coefficient for shrinkage from setting of cement mortar in air is given as about 0.0010. Mr. Rafter* found for cement (1 : 2 to 1 : 4) mortar weighing from 119 to 128 lbs. per cu. ft. and sand-

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* *Trans. Am. Soc. C. E.*, Vol. XLII, p. 152.

stone weighing 155 lbs. per cu. ft., the concrete formed varied in weight from 139 to 148 lbs. per cu. ft., the volume of mortar varying from 33% to 40% of the broken stone.

40. *What rise and fall of temperature shall be allowed?* Steel arches readily take the temperature of the air and are quickly heated by the sun, so that the extreme fall and an equal rise, at least, above the mean should be allowed.

For concrete, stone or brick, it seems certain that no such extremes are experienced, otherwise many of the hundreds of concrete bridges built in recent years would show cracks, for many of them are built of very light section. Concrete, stone, brick, and earth are very poor conductors, and where the arch is covered with earth, doubtless its top does not vary a great deal in temperature. The same may be said for the lower side of arch, as it is next the water and screened from the direct rays of the sun. As concrete or masonry bridges are usually fin-

ished in the warmer months, the mean temperature of 53° F. was assumed in Art. 36, and a rise and fall of 40° and 63° respectively. Doubtless half of these quantities would be nearer the truth for the changes in temperature of the arch ring, if 40° and 63° rise and fall referred to the air, and the amounts should decrease as the depth of arch or span increases, though, of course, experiment alone can decide the whole matter.

The exact determination of the maximum stresses in concrete arches is complicated too by other considerations: the shrinkage of the concrete and permanent set under stress.* The shrinkage should not prove so great, if the concrete is well rammed, still there will always be some shrinkage which will have the effect of taking off some of the computed stress from the concrete and adding more to the steel ribs.

* See Mr. Molitor's paper on Three-Hinged Masonry Arches, in *Trans. Am. Soc. C. E.*, Vol. XL, p. 56, not only on this point, but for a complete discussion of the three-hinged concrete arch.

It may be thought that increasing the depth at the crown may be the remedy for temperature and allied stresses, but as this increases the horizontal thrust, the tensile stresses are not greatly altered.

It is suggested that more steel should be used than strict theory requires, with the ribs spaced closer together, to make the practice conform more to the hypothesis. This increase can be made to satisfy the condition, at any critical point in the arch, that all the bending moment due to loads and temperature should be borne entirely by the steel ribs at some stress under the elastic limit—say 20,000 pounds.

It is very evident from the foregoing that the steel concrete arch is, in every way, superior to the concrete arch, as the unit stress actually sustained by the steel is small, and it thus furnishes a reserve of strength which may be called for sometime under exceptional loading.

CHANGE OF SPAN.

41. If the abutments give a small amount, the span of the neutral line increases a small amount d. On replacing $l\varepsilon t^{0}$ in the formulas above by d, the full H is found as before. The effect is equivalent to a fall of temperature which requires the unstrained arch to be fitted to a larger span. The result is the same for the elastic shortening of the arch due to the tangential forces T, hitherto neglected. The sum of the horizontal projections of the shortening of each part s of the neutral line, due to T, which can be computed, is the change of span, and replaces $l\varepsilon t^{0}$ above.

If the stress f, in pounds per square foot, due to T alone, is taken roughly as the same on each cross-section of the arch, the change of the span of the neutral line is $\left(\frac{f}{\mathrm{E}}l\right)$ where E is the modulus in pounds per square foot, and l the span of the neutral line : f might here be found as a rude average for the whole arch.

ARCH HINGED AT ENDS ONLY.

42. For an *arch hinged at the ends only*, the reactions pass through the centre of the hinges, neglecting any friction there. As the span is invariable, the condition to be fulfilled by the equilibrium polygon is given by Eq. (11), Art. 12, or putting $I = I_1 + nI_2$ for brevity,

$$\Sigma \frac{Mys}{I} = 0,$$

where M, I_1, I_2, and y are taken at the middle of the corresponding s, and the summation covers the entire span. Two solutions may be given: In the first, suppose the neutral line to pass through the hinges A, B, and that AB is horizontal. Divide the neutral line of the arch ring into equal parts, each of length s, and measure on a drawing the vertical ordinates y from AB to the centre a of each division, and estimate $I = I_1 + nI_2$ at each point a. Next, draw, with a trial pole, an equilibrium polygon due

to the loads, through A and B. Thus, through any point A^1 in the vertical through A, construct an equilibrium polygon and suppose that it meets the vertical through B at B^1. The ordinates through the points a from A^1B^1 to the polygon are the values of y^1 below, which can now be *supposed* laid off from AB to give the required polygon passing through A and B. It need not be actually drawn. For the greatest accuracy, the successive weights should be taken as the weights of arch and load between the successive a',s (as in Fig. 6).

Then the preceding condition reduces to

$$\Sigma \frac{(y^1 - y)y}{I} = 0, \therefore \Sigma \frac{y^1 y}{I} = \Sigma \frac{y^2}{I},$$

since $M = H(y_1 - y)$, where H is the horizontal thrust. Both members of the last equality can be estimated. If the equality does not obtain, alter all the ordinates of the type y^1 in the ratio

$$\Sigma\left(\frac{y^2}{\mathrm{I}}\right) \div \Sigma\left(\frac{y^1y}{\mathrm{I}}\right),$$

to locate the points of the true equilibrium polygon satisfying the conditions. The true horizontal thrust is equal to the trial thrust multiplied by

$$\Sigma\left(\frac{y^1y}{\mathrm{I}}\right) \div \Sigma\left(\frac{y^2}{\mathrm{I}}\right).$$

The second solution requires the division of the neutral axis to be made as in Art. 14, thus leading to the simple condition,

$$\Sigma \mathrm{M}y = 0 \text{ or } \Sigma y^1y = \Sigma y^2,$$

and the solution proceeds as above.

Also see "Theory of Solid and Braced Elastic Arches" for I constant and for temperature stresses.

ABUTMENTS.

43. Lack of space prevents an extended treatment of *abutments*. The reader may consult for partly empirical formulas

Trautwine's "Pocket Book," Dubosque's "Ponts et Viaducts," or Van Nostrand's Magazine for Dec. 1883. The resultant acting on any joint of the abutment is at once found by combining the ascertained thrust of the arch with the weight of abutment and load over the joint. It should pass within the middle third. To avoid sliding it is well to construct the courses perpendicular to the resultant.

THE SPANDREL RESISTANCE FOR VOUSSOIR ARCHES.

44. It will prove interesting to give a construction for spandrel resistance at this point.* It is found by assuming that the resultants on the successive joints or sections, are tangent to the centre line of the arch ring. Thus in the full centre arch of Fig. 7, lay off, on the extreme left vertical upwards, the loads from the crown to each joint in turn. Thus 0 — 11

* See Rankine's *Civil Engineering*, Art. 138.

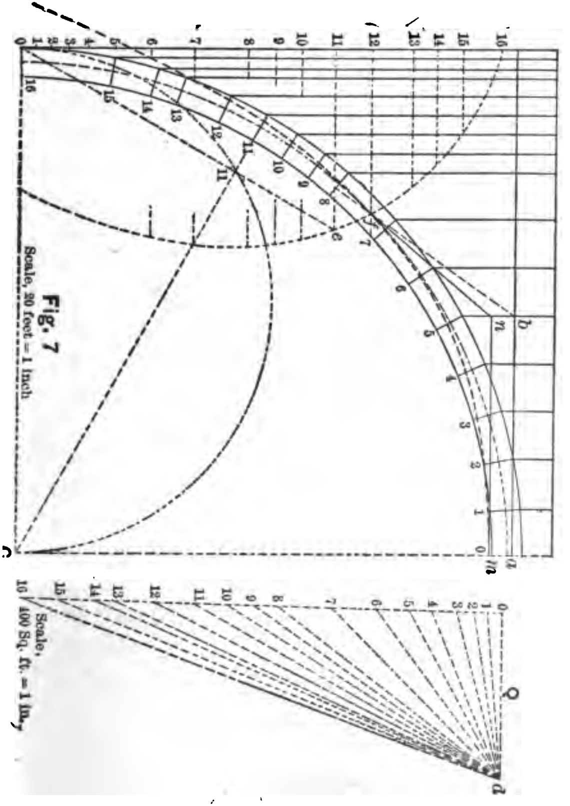

Fig. 7

Scale, 20 feet = 1 inch

Scale, 400 Sq. ft. = 1 in.

is the load (to scale) of arch and spandrel from the crown to joint 11. Draw $\overline{oe}$ parallel to the centre line at joint 11—the construction shown effects this easily, by drawing $\overline{o,11}$ through the intersection of $\overline{c,11}$ with the semi-circle. Next draw a horizontal line through 11 on the scale of loads, to the intersection e with the line just drawn; then $\overline{oe}$ represents the magnitude and direction of the resultant at joint 11, whose two components $\overline{o,11}$ and $\overline{e,11}$ are respectively, the load from the crown to joint 11, and the *total horizontal thrust exerted below* joint 11.

On repeating this construction for each joint, we find the horizontal thrust exerted below each joint; the horizontal thrust, then exerted upon a single voussoir, as that between joints 11 and 12, by the spandrel, is thus the difference between the line $\overline{e,11}$ and the horizontal $\overline{f,12}$. At joint 8, called "*the joint of rupture*," the horizontal thrust obtains its maximum; and above this point, in this case, the spandrel would have to

exert tensile forces to cause the centre line to become the true line of pressures; but if it cannot do this, the horizontal thrust from joint 8 to the crown is constant.

In the latter case, the tangent to the centre line at joint 8 is produced to intersection *n* with the vertical through the centre of gravity of the load from the crown to joint 8, and from this point *nm* is drawn horizontal to intersection *m* with the crown joint. From this centre of pressure, the line of the centres of pressure is drawn as usual, as shown by the dotted line down to joint 8, from whence it follows the centre line as assumed. (Of course this is not the true curve, as we shall see further on.)

45. This tacitly assumes incompressible spandrels, for as an actual semi-circular arch, on striking the centres, tends to spread outwards at the haunches, if this tendency is *entirely* prevented by the spandrels, this can only happen with incompressible spandrels.

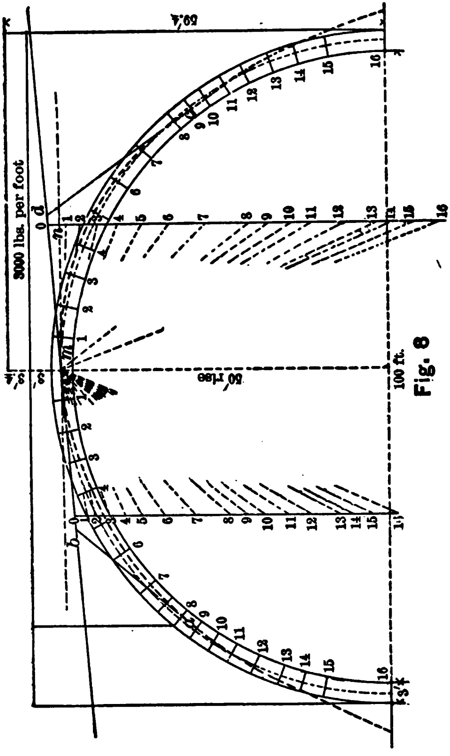

Fig. 8

To illustrate, in Fig. 8 is drawn a line of resistance (not the true one) for the arch subjected to an eccentric load on the supposition that no spandrel resistances are experienced.

This curve or any other that may be drawn (including the true one) would pass near or outside of the extrados, at joints 16, so that the arch would tend to rotate outwards about these joints and thus spread at the haunches. If the spandrels really form an extension of the abutment upwards with equal solidity of construction, especially if they are bonded with the voussoirs, as is usual, their action is approximately as indicated above, and as a basis for computation, the arch and spandrel from joints 8 down, with the abutments proper, can be regarded as an immovable abutment, and the part of the arch above joint 8 can be treated as an elastic arch by preceding methods. The equilibrium polygon for this upper part of the arch is then found as in Art. 18, et seq., if it nowhere leaves the middle

third.* It of course does not agree with the dotted curve from m down to centre of joint 8, but lies on either side of the centre line. At joint 8 the true centre of pressure lies below the centre of that joint. Below that, the spandrels exert just so much resistance as to cause the line of the centres of pressure to keep near the centre line, so that no appreciable spreading occurs. These resistances could easily be estimated by a method somewhat analogous to that given in Art. 43, but the results, even when known, are of but little importance.

A better disposition of the masonry would be to omit the spandrels and to increase the radial depth of the arch ring from joint 8 (about) to the abutments, so that the true line of the centres of pressure, should everywhere be contained in the middle third for the loading assumed.

This plan is actually cheaper for a con-

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* Also see the author's "Theory of Voussoir Arches," 2d edition.

crete arch. If spandrels were used here, and the whole, from crown to abutment, was treated as a solid arch, after the method of Art. 18, etc., it would be found, as in the example treated there, that the part from joint 8 down, exercises but little influence upon the curve of pressure, so that it can practically be treated as a part of the abutment.

METHODS OF FAILURE OF ARCHES.

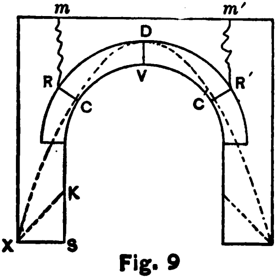

Fig. 9

46. In Fig. 9 is shown the method of failure of semi-circular arches when the abutments are too narrow : the crown

sinks and the joint opens at the intrados at V and at the extrados at R and R', the spandrels crack above R and R', and a portion mRCKX of spandrel, arch and abutment, detaches itself, rotating about X. A portion XKS of the abutment is left standing when the mortar is weak.*

In Van Nostrand's Magazine for March, 1873, page 193, is described the Pont-y-Tu-Prydd arch bridge of 140 ft. span, 35 ft. rise and only 1 ft. 6 in. depth of rubble arch ring in the body of the arch! The curve is an arc of a circle. On striking the centres, the arch fell, the weight of the haunches forcing up the crown. The mason, noticing this, lightened the spandrels with cylindrical openings, filling the space between with charcoal, and succeeded in making the bridge stand. The resistance line is found to be everywhere contained within the arch ring and the bridge stands to this day. It may not only prove instructive as a precedent to

* See other methods of failure of arches given in "Theory of Voussoir Arches, 2d ed., Art. 23 and Appendix.

be avoided as regards the very thin arch ring, but it illustrates the only method of failure of a segmental arch with immovable abutments (when crushing of the voussoirs is not in question) and shows besides, that the stability of the segmental arch is increased by lightening the spandrels. This lightening the load over the haunches can be effected in various ways, as by substituting walls parallel or perpendicular to the outer spandrel walls connected by flags or arches, for the solid backing, or using a steel framework for the interior walls, etc. An approximate construction will give at once the height of surcharge (supposed homogeneous and and of same density as the voussoirs) in order that the centre line of arch ring may be the true equilibrium curve.

HEIGHT OF SURCHARGE FOR EQUILIBRIUM.

47. In Fig. 10, divide the half span of centre line into equal parts, and erect verticals meeting the centre line of arch ring

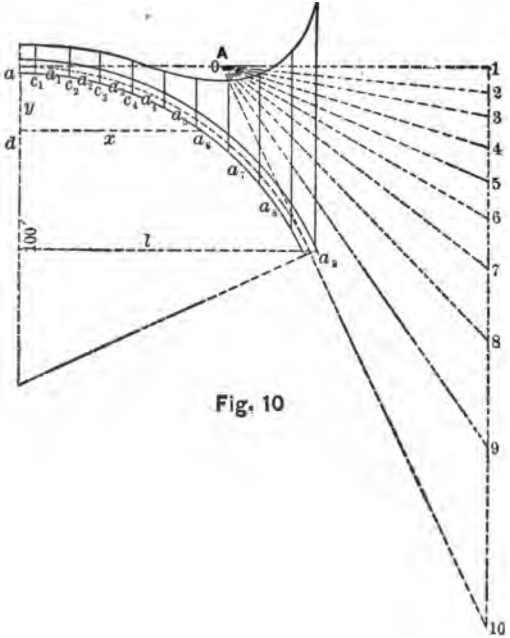

Fig. 10

in a_1, a_2 '. . . ; call the mid points of aa_1, a_1a_2, . . . , c_1, c_2, . . . , respectively and consider $ac_1c_2c_3$, . . . , the equilibrium polygon for the arch. From some point A, draw A1 horizontal, A2 ∥ c_1c_2, A3 ∥

c_2c_3, . . ., and draw the vertical 1 2 3 . . ., such a distance to the right of A, that the intercepted length 1 2 is equal to the median (vertical from soffit to top of surcharge) at c_1 or depth at crown nearly. Then the length of medians at c_2, c_3, . . ., are 2 3, 3 4, . . ., respectively and the load acting at c_1 or c_2, . . ., is thus equal to the corresponding median, multiplied by horizontal projection of aa_1, or a_1a_2, . . ., multiplied by unit of weight.

The walls, arches, charcoal, etc., can now be designed to give these weights at the respective points, in which case the centre line is the eqnilibrium polygon for dead load, assuming all loads to act vertically. The latter hypothesis is nearer the truth when *cross* walls are used.

48. If the cross walls are at other points than those marked c, the weight of any one with its load, must equal the median along its vertical centre line, multiplied by the sum of the horizontal distances from this median to verticals drawn midway between the vertical axes of adjacent

cross walls and this product then multiplied by the weight per unit. This assumes the cross walls spaced equal horizontal distances apart.

When longitudinal walls are used, their whole weight first rests upon the arch; but if the arch sinks appreciably, a small amount may be carried directly to the abutment, through cantilever action aided by friction. The theory of vertical loads would then seem to be admissible. But an important fact must be noted, i.e., that these walls resist spreading of the arch at the haunches and thus add materially to the stability for arches of large central angle.

Compacted earth filling can furnish little or no active horizontal thrust, but it likewise adds to the stability by resisting appreciably any tendency to spread at the haunches.

49. If the centre line is a parabola, it is easy to show that the upper limit of the surcharge must be everywhere the same vertical distance from the soffit; so

that where the surcharge is very high the parabola is the best form of arch ring.

This may be shown analytically, in an easy manner. Thus conceive $a_1, a_2, \ldots a_9$, Fig. 10, to consist of a thin metallic ring, that is to sustain a uniform horizontal load, w per foot, without bending; required the form of the curve $a \ldots a_9$. It is necessary that the line of pressures coincides throughout with the rib, for if it departs from it at any point, the resultant for that point multiplied by its lever arm to that point, gives a bending moment M, which the thin rib is supposed incapable of resisting.

Let $\overline{aA}$ be the axis of X, the vertical down from a the axis of Y. The resultant at a is the horizontal thrust Q. Take moments of this force, and the downward acting weight on the part aa_6, about a_6, whose coördinates are y and x,

$$\therefore \quad M = Qy - \frac{wx^2}{2}.$$

Now M must equal zero for every point of the arch, in which case *the line of pressures will coincide with the figure of the rib.*

Placing M = 0, we deduce,

$$x^2 = \frac{2Q}{w} y,$$

the equation of a parabola, Q.E.D.

CHAPTER II.

UNDERGROUND ARCHES.

FORMULAS FOR PASSING A LINE OF RESISTANCE THROUGH THREE GIVEN POINTS.

50. In the arch ADB, Fig. 11, suppose it required to pass a line of resistance through the points A, E and B.

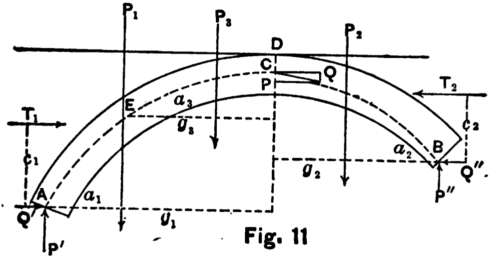

Fig. 11

Let C be another point of this curve, at the crown, where the horizontal component of the pressure is Q, the vertical component P. Call the vertical compo-

nents of the loads on the segments AD, DB and ED, P_1, P_2, P_3, respectively; their *horizontal* components, T_1, T_2, T_3, respectively.

Call the perpendicular distances from P_1 and T_1 to A, a_1 and c_1; from P_2 and T_2 to B, a_2 and c_2; and from P_3 and T_3 to E, a_3 and c_3, respectively.

Also call the vertical distances of C, the point of application of the inclined thrust at the crown, above A, B and E, b_1, b_2 and b_3, respectively; and the horizontal distances of the same points, A, B, E, from the crown, g_1, g_2 and g_3.

51. We now take moments in turn about A, B and E. In Eqs. (1) and (3), we suppose the arch to the right of the crown removed, and its effect replaced by the resultant of P and Q acting to the left, P being + when acting upwards; in Eq. (2), the part left of the crown is supposed removed, and a force equal and directly opposed to the resultant of P and Q acting to the right.

We thus find :

$$a_1P_1 - g_1P + c_1T_1 = b_1Q \qquad (1)$$

$$a_2P_2 + g_2P + c_2T_2 = b_2Q \qquad (2)$$

$$a_3P_3 - g_3P + c_3T_3 = b_3Q \qquad (3)$$

Equating the values of Q in (1) and (2), we find,

$$P = \frac{b_2(a_1P_1 + c_1T_1) - b_1(a_2P_2 + c_2T_2)}{b_2g_1 + b_1g_2} \qquad (4)$$

From (1) we obtain,

$$Q = \frac{a_1P_1 - g_1P + c_1T_1}{b_1} \qquad (5)$$

These equations suffice to determine P and Q, when the position of C is known. When, however, we can only locate the points A, E and B, the values of P and Q and the position of C is found as follows :

For convenience let us make the following abbreviations :

$$g_2 + g_3 = d_1,\ \ g_1 - g_3 = d_2,\ \ g_1 + g_2 = d_3,$$
$$b_2 - b_3 = e_1,\ \ b_1 - b_3 = e_2,\ \ b_1 - b_2 = e_3.$$

Now subtract (2) from (1),

$$a_1P_1 - a_2P_2 - Pd_3 + c_1T_1 - c_2T_2 = Qe_3. \quad (6)$$

also, subtract (3) from (1),

$$a_1P_1 - a_3P_3 - Pd_2 + c_1T_1 - c_3T_3 = Qe_2.$$

Equating the values of Q drawn from these last two equations, and noting that $a_1P_1(e_2 - e_3) = a_1P_1e_1$; $c_1T_1(e_2 - e_3) = c_1T_1e_1$ we have,

$$P = \frac{e_1(a_1P_1 + c_1T_1) - e_2(a_2P_2 + c_2T_2) + e_3(a_3P_3 + c_3T_3)}{e_2d_3 - e_3d_2} \quad (7)$$

Substituting in Eq. (6) the value of P just found, reducing the terms of one member to the same denominator, collecting like terms, whose coefficients are of the type *ed*, and noting that $e_2 - e_1 = e_3$ and $d_3 - d_2 = d_1$, we have,

$$Q = \frac{d_1(a_1P_1 + c_1T_1) + d_2(a_2P_2 + c_2T_2) - d_3(a_3P_3 + c_3T_3)}{e_2d_3 - e_3d_2} \quad (8)$$

From (1) we have,

$$b_1 = \frac{a_1 P_1 - g_1 P + c_1 T_1}{Q} \qquad (9)$$

to fix the position of C at the crown.*

We have always for the reactions at A and B, $P' = P_1 - P$, $P'' = P_2 + P$, $Q' = Q - T_1$, $Q'' = Q - T_2$.

52. The above equations apply directly to unsymmetrical arches, solicited only by vertical forces by making T_1, T_2, and T_3 zero.

When the arch and load is symmetrical, $P = O$. If the point of application at the crown is known, we have from (5),

$$Q = \frac{a_1 P_1 + c_1 T_1}{b_1} \qquad (10)$$

If two points A and E are given, we have then $g_1 = g_2$, $h_1 = h_2$, $d_3 = 2g_1$, $e_3 = o$, $P_1 = P_2$, $T_1 = T_2$, $a_1 = a_2$, $c_1 = c_2$; whence from (8),

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* A graphical construction can be given for the general case, but is omitted for want of space.

$$Q = \frac{a_1P_1 + c_1T_1 - (a_2P_2 + c_3T_3)}{e_3} \quad (11)$$

The position of Q is then found from (9) by making $P = o$.

Eq. (11) is very easily deduced independently.

APPLICATION TO CULVERTS.

53. Let Fig. 12 represent a culvert having no masonry backing, with the embankment above it partially completed, so that when the material of the embankment is reduced to the same specific gravity as that of the arch, a line $\overline{ai}$ (taken straight for simplicity) will limit its top; the earth being level to the left of a and to the right of i.

The tables for the *vertical forces* are made out as usual.* The mean heights of the trapezoids are represented by the dotted lines and the sum of the first three trapezoids will be considered as the sur-

* See the author's "Theory of Voussoir Arches."

face from the crown to the third joint; similarly for other joints.

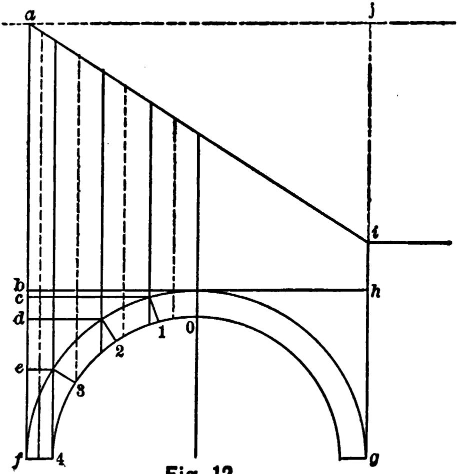

Fig. 12

The horizontal forces are due to the earth pressure and are very difficult to estimate exactly, In a mass of earth with an *unlimited level surface*, the hori-

zontal pressure per square unit at a depth x,*

$$p = wx \frac{1 - \sin.\ \Phi}{1 + \sin.\ \Phi} = wx\ tan^2 (45° - \tfrac{1}{2}\Phi).$$

When the upper surface is at the angle of repose Φ, the pressure per square unit of a vertical plane parallel to the slope, is,

$$p' = wx \cos.\ \Phi,$$

w represents the weight per cubic unit of the earth.

These formulæ are modified, when the earth is not of unlimited extent, the friction of the abutting surfaces causing a change in the direction of the pressure.

As the surface above is sloping from a to i, and level elsewhere, only a rough approximation to the pressure is possible.

Cohesion, likewise, plays an important *role* in earth pressure; its influence becoming much more marked as the em-

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* See Rankine's "Civil Engineering," p. 322; also Van Nostrand's Science Series, No. 3, pp. 99, 100.

bankment grows older. For new embankments it is well to neglect it.

Let us assume, as a rough approximation, that the horizontal pressures, due to the earth, on voussoirs 1, 2, 3 and 4, correspond to the heights x measured along the dotted lines from the extrados of each voussoir to the surface ai, and are given by formula for p above.

The surfaces against which these pressures act for voussoirs 1, 2, 3, 4, are, $\overline{bc}$, $\overline{cd}$, $\overline{de}$, $\overline{ef}$, respectively; so that the horizontal pressure acting on the third voussoir, for instance, is equal to the product of the height $\overline{de}$, by the height of the surcharge from the extrados to the surface, by $tan^2\ (45-\frac{1}{2}\Phi)$, ($w$ being taken as unity). In the following examples let $\Phi = 30°$, so that, $tan^2\ (45-\frac{1}{2}\Phi) = \frac{1}{3}$.

The horizontal pressure then upon the third voussoir is, $\overline{de} \times x \times \frac{1}{3}$. It may be written $\frac{yx}{3}$ for any voussoir. The lever arms of these forces, about the top of the arch, are the vertical distances from the

line $\overline{bh}$ to the middle of the segment $\overline{bc}$, $\overline{cd}$, $\overline{de}$ and $\overline{ef}$. The sum of the moments of these forces, down to any joint, divided by the sum of the same forces, gives the vertical distance from the line $\overline{bh}$ to the resultant of the forces taken.

54. *Example.*—Let the span of the semi-circular culvert, Fig. 13, be 11.30 units of length, the depth of voussoir 0.94, the height $\overline{ab}$ of the reduced surcharge 25.12, and the height $\overline{hi}$, 12.56. If the backing was solid up to $\overline{bh}$, the horizontal forces would be due more nearly to the depth from a to the voussoirs on the left, and from i on the right.

Now let it be required to pass a curve of resistance, 0.41, below the top of the crown joint, and through the lower middle third limits, at joints six on either side.

The vertical loads from the crown to joints six on left and right are $P_1 = 133.23$, $P_2 = 101.28$; the distances of their resultants from the verticals through the crown are 3.01 and 2.79 respectively,

whence by measurement on the drawing, $g_1 = g_2 = 5.1$ and $a_1 = 5.1 - 3 = 2.1$, $a_2 = 5.1 - 2.8 = 2.3$.

Similarly,

$$T_1 = 26.41,\ c_1 = 3.5 - 1.7 = 1.8$$
$$T_2 = 18.51,\ c_2 = 3.5 - 1.6 = 1.9.$$

whence, by Eq. (4),

$$P = \frac{a_1 P_1 + c_1 T_1 - (a_2 P_2 + c_2 T_2)}{2g_1} = 5.8$$

Also, by Eq. (5), $Q = 96$.

Now lay off on vertical lines, $\overline{08}$, to left and right of the centre, the vertical loads from the crown to the joints in order. Also lay off the distances, on the horizontal through the summit, from the crown to the centres of gravity of the total vertical loads in order. Thus $\overline{S6} = 3.01$, corresponding to $P_1 = 133.23$.

Next, lay off on the horizontals through 1, 2, . . . , to 1′, 2′, . . . , the total horizontal forces for the left and right side respectively. Also lay off on vertical lines their

points of application. Thus the total horizontal earth thrust from the crown to joint 6 on the left is $T_1 = 26.41$; and its point of application is $\overline{g6} = 1.71$ below the summit. To find the thrust at the crown, lay off $\overline{mn} = Q$ horizontally, and $\overline{no} = P$ vertically downwards: $\overline{mo}$ ($= vo$ drawn ‖ mo) is then the resultant at the crown joint in position and magnitude. Now, to find the centre of pressure on a joint, as the 6th on the left, draw vertical and horizontal lines $\overline{6b}$, $\overline{6b}$, through the points of application of P_1 and T_1, to intersection b; which is thus the point of application of the resultant of P_1 and T_1, represented by a straight line from 0 to $6'$ in the force polygon on the left. From b draw ba ‖ $\overline{06'}$ to intersection a with $\overline{mo}$ produced; from a draw $\overline{ac}$ ‖ $\overline{v6'}$ to intersection with joint 6 at its centre of pressure. It is evident that the resultant there is represented by the line $\overline{v6'}$, the resultant of $\overline{06}$, $\overline{66'}$ and $\overline{vo}$, or of P_1, T_1, and the inclined thrust at the crown;

similarly on the right side, to find the position of the resultant on joint 8, we find d, 3.29 to the right of S and 3.34 below it; thence draw $de \parallel \overline{08'}$ to intersection e with $\overline{mo}$ produced; thence draw $\overline{ef} \parallel \overline{v8'}$ to f the required point; the magnitude and direction of the resultant being represented there by the line $\overline{v8'}$.

The line of the centres of pressure thus found, represented by the dotted line, leaves the middle third at joints 4, 5 and 8 on the right, and at joint 8 on the left.

A line nearer the truth, if the passive resistance of the earth is still neglected, would probably have m raised somewhat and f nearer the extrados.

55. The earth is next supposed *level at top*, the distance $ba = hj$, Fig. 12, being 25.12 units. On making out vertical and horizontal forces as before, for one side only, we find from Eq. (10), $Q =$ 122.2 cubic units of stone, on passing a curve of resistance through the upper middle third limit at the crown and the lower middle third limit at joint 6.

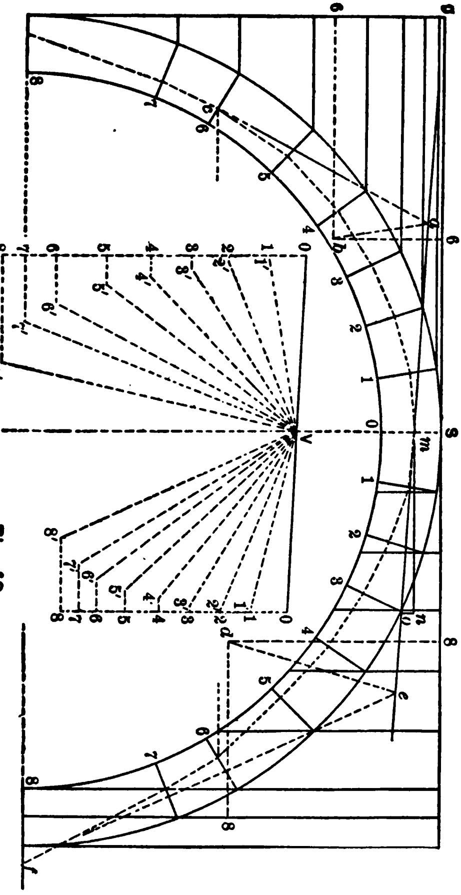

Fig. 13

The curve thus found keeps everywhere in the middle third except at joints 8, where it nearly reaches the extrados.

56. If the arch stones are not increased in depth near the abutment, joints 8 will tend to open at the intrados; but this they cannot do unless the haunches spread; which is in turn resisted by the spandrels; or if there are none, by an increased horizontal thrust which the earth is capable of putting forth, thus keeping the line of resistance within the arch ring, e.g., within the middle third, if the deformation that the earth permits is small.

Experience shows that very thin arch rings, built in rubble, often can fulfill the conditions of stability when embanked over carefully; the centres being struck after the embankment is mostly completed.*

The theory of earth pressure† demon-

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* See Trautwine's "Engineer's Pocket Book," p. 347.

† See Van Nostrand's Science Series, No. 8, p. 46.

strates that the earth is capable of exerting much greater passive resistances than active thrusts, but their exact amounts are uncertain; though if a line of centres of pressure is assumed from about joints 5 down, to lie near the centre line, their magnitudes can be found after the method of Art. 43. If the arch stones are increased in size near the abutment, so that a line of centres of pressure can everywhere be inscribed in the middle third, the arch will need but little aid from the passive resistance of the earth, and this precaution is strongly urged.

57. The dimensions of the preceding culvert and surcharge may be taken in any unit, as feet, meters, etc.

If the unit taken is the meter, it corresponds to a railroad culvert at Schwelm, the top of the embankment being $31.^{m}40$ above the top of the arch, corresponding to a weight $25.^{m}12$ high of materials as dense as the voussoirs, as given by Scheffler.

The normal pressures on joints 6 or 8 are

about 41.5 tons per square foot, if uniformly distributed, and 83 tons when the pressure acts at the middle third limit. Granite or limestone has a crushing weight of 400 to 500 tons per square foot when of good quality, which was not the case here, and a large number of voussoirs were crushed in various parts of the arch in consequence. If the unit of the drawing, Fig. 13, is the foot, the uniformly distributed normal pressure per square foot, is 11.3 tons, and if the centre of pressure on the critical joint was at the middle third limit, the stresses would be 22.6 tons per square foot at the intrados, which sandstone and best brick are able to sustain.

In the case of very high embankments, the full height of the surcharge does not bear on the culvert or tunnel arch, part of it being carried by friction and a natural arch action to the sides.*

In designing the *abutment* for a cul-

* See Van Nostrand's Science Series. No· 3, pp. 11, 12,

vert, it is well not to count on this action, and since the ground generally rises abruptly from the foundation, it is advisable not to count on any horizontal earth thrust against the back of the abutment unless the earth is well rammed. The thrust of the arch on the abutment can be found approximately as in Art. 54, taking the centres of pressure at the crown and haunches (about joint 5, Fig. 13), at the centre of the joints for additional safety.

TUNNEL ARCHES.

58. The indetermination as to the real acting forces is much more pronounced for tunnels, so that experience has to be resorted to. Rankine gives the following formula, founded on practice, for the minimum thickness, t of tunnel arches,

$$t = \sqrt{.12\,r}, \quad r = \frac{a^2}{b}:$$

where $a =$ rise and $b =$ half span.

"This is applicable where the ground

is of the firmest and safest kind. In soft and slippery materials, the thickness ranges from $\sqrt{.27r}$ to $\sqrt{.48r}$."

The *arch* is peculiarly adapted for a tunnel support; for it is the great advantage of the arch, that it *will not be forced in* in one place *without it is forced out* at another. The latter the enveloping mass generally prevents, if stones and earth are packed in tight back of the arch; so that the arch so constructed should generally stand unless crushed from a too heavy load.

As, in practice, tunnel arches are not thus crushed, we may infer, as stated before on theoretical grounds, that only a part of the superincumbent material presses on them. In every deep tunnel, the thickness of the arch ring is not increased over that due to a comparatively small height, as is inferred from the preceding formula.

If a quicksand is encountered on one side or the other, the curvature of the arch must be sharply increased there, or

the arch may be forced in, as has happened in certain treacherous clays.

59. Assuming the preceding formulas for earth thrust and the depth of surcharge that is supposed to press, in accordance with this theory, the stability of a tunnel arch is investigated as previously explained in the case of culverts.

Thus take the tunnel arch under the Thames, Fig. 14, whose dimensions in feet are as follows : the thickness of the arch ring is about 3, the radius of the upper part $\overline{0..6}$ is 7; and of the inferior part $\overline{6..10}$, 28.25 feet; the upper part being formed of three concentric rolls without bond, but supposed to act as one mass.

The earth and water above the tunnel is supposed to exercise upon the arch a pressure corresponding to a load 24.75 ft. high of material like that of the voussoirs; the reduced surcharge being supposed level at top for simplicity.

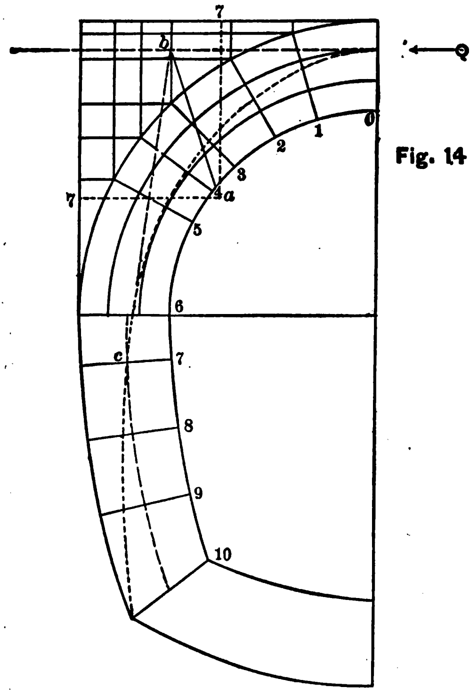

Fig. 14

One line of resistance drawn is represented by the dotted line, and leaves the inner third of the arch ring at joints 9

and 10. However, if the earth is packed tight around the arch, joint 10 cannot open on the inside as the haunches cannot spread, and in fact, the induced (passive) resistance of the earth, giving greater horizontal forces than assumed, will cause the centres of pressure below *c* to lie nearer the centre of the joints. It is better thus than to have the centre line coincide with the dotted curve throughout; for then, if the active earth thrusts are actually greater than computed the arch ring may be forced in somewhere. We conclude that this arch is well designed as to form.

When a tunnel is constructed through firm earth, the active horizontal earth thrusts are small and even nil in some cases, so that the portion from joints 6 to 10 may be made vertical. This should not be done if the material is a kind of clay that swells when exposed to air and moisture. In Fig. 14, the inverted arch at the bottom is intended to prevent the forcing in of the sides.

THE ELLIPSE THE PROPER FORM FOR A TUNNEL ARCH.

60. The resistance line in Fig. 14, resembles a portion of an ellipse.

We can easily prove analytically that the ellipse is the proper form for a tunnel arch, when the depth of surcharge is so great that the in-

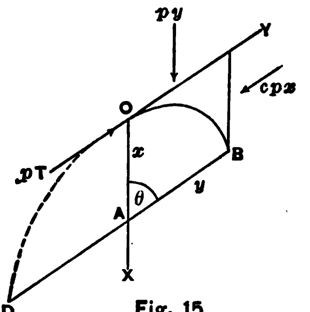

Fig. 15

tensity of the thrust of the earth at any part of the arch is practically the same. To take the most general case, let the top surface of the earth be ∥ OY in the adjoining Figure. Call the pressure, per unit of inclined plane OY, in a vertical direction, p; the conjugate pressure

in a direction ‖ OY, per unit of a vertical plane OA, can be represented, according to the theory of earth pressure, by cp, c being a constant. Let $OA = x$, $AB = y$, $OAB = \theta$, and call the thrust at O in the direction OY, tangent to the rib BOD at O, pT. This rib, or "linear arch," is not supposed to have any bending moment at any point, otherwise a deformation would ensue, The total vertical pressure on OB is py, the con. jugate pressure is cpx. Being uniformly distributed, their lever arms about B are

$$\frac{y \sin. \theta}{2}, \quad \frac{x \sin. \theta}{2} \text{ respectively.}$$

Now, if any point, as B, of the arc is to be a point in the line of pressures, we must have, taking moments about B,

$$p\mathrm{T}x \sin. \theta = \left(\frac{py^2}{2} + \frac{cpx^2}{2}\right) \sin. \theta,$$

$$\therefore \quad y^2 = 2\mathrm{T}x - cx^2,$$

the equation of an ellipse,

Q.E.D.

The equation of an ellipse referred to a diameter and the tangent at its vertex, a and b being the semi-conjugate diameters, is

$$y^2 = \frac{b^2}{a^2}(2ax - \underset{x}{\cancel{w}}^2).$$

Comparing with the above, we have,

$$T = \frac{b^2}{a}, \quad c = \frac{cp}{p} = \frac{b^2}{a^2}.$$

Or, *The intensities of the conjugate pressures are as the squares of the diameters to which they are parallel.*

If in the equation above we make, $x = OA = a$, we find, $y = AB = b$; whence from the last equation,

$$\frac{a.\ cp}{b.\ p} = \frac{b}{a}.$$

Now the thrust at $O = pT = acp$, whilst that at the ends of the conjugate diameter DB, acting ∥ OX, is bp; hence, *these forces are proportional to the diameters to which they are parallel.*

To construct the arch, c and a or b must be given to find the other semi-diameter from the equation, $c = b^2 \div a^2$.

When the Top Surface of Earth is Level, OY becomes level, and $\theta = 90°$. a and b are now the semi-axes of the ellipse. From the theory of earth pressure, $\frac{cp}{p} = \tan.^2 (45° - \frac{1}{2}\theta)$, whence, for a tunnel arch,

$$\frac{\text{horizontal semi-axis}}{\text{vertical semi-axis}} = \frac{b}{a} = \tan. (45° - \tfrac{1}{2}\theta);$$

θ being the angle of repose.

Next, make $c = 1$ (θ being $90°$) and the ellipse becomes a *circle*. At any point of the circle consider the two equal forces, p, p, at right angles and acting on a unit of area of planes $\perp$ to them. Their resultant acts normally to the circle, and its intensity is easily found to be p, the intensity of the vertical and horizontal components.

Calling $r = a = b$, the radius of the circle, we have the thrust at $\mathrm{O} = p\mathrm{T} = pr$, or *the product of the intensity on a unit of circumference by the radius*. This thrust is the same all around the ring. The above are the principal deductions by Rankine for the arches considered above, but the proof is entirely different to that given by him. The last formula is sometimes used to find the horizontal thrust at the crown for a symmetrical arch uniformly loaded, as the pressure there is normal to the equilibrium curve, and its intensity p is equal to the depth of arch and reduced surcharge at the crown, multiplied by the weight of stone per cubic unit. As the radius r of the linear arch at the crown, is not known, it may be assumed equal to that of the intrados there, and the approximate thrust $= pr$ computed.

Du Bosque uses this formula of Navier's for the thrust, in dealing with the arches of certain proportions examined by him, but, of course, for

other proportions it would not work. Sometimes a rough approximation to the thrust is all that is wanted; in that case, the formula is serviceable.

CHAPTER III.

GROINED AND CLOISTERED ARCHES.

61. Groined and cloistered arches are formed by the intersection of two cylindrical arches, having the same rise and their axes in the same plane. The groined arch is formed by removing those portions of each cylinder which lie under the other and between their common curves of intersection; the cloistered arch by removing the portions of each cylinder above the other and exterior to their common intersection.

The forces exerted in any part of a groined arch of masonry are best shown by an example.

GROINED ARCH.

62. Let the square ABCD, Fig. 16, be the plan of a groined arch, AC and BD representing the groins; the elevation is

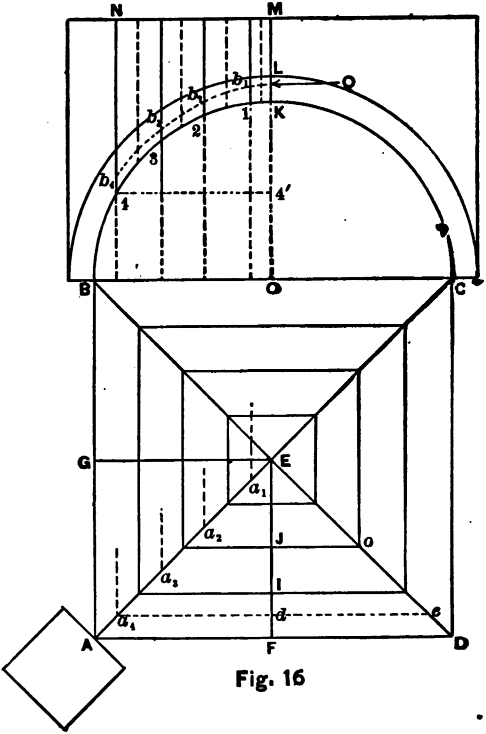

Fig. 16

shown, at BMC of the front face AD. There are abutments at A, B, C and D, one of which is shown at A in plan.

Let us divide the portion of the arch and load between the groins into simple (cylindrical) arches, as AID, a_1Io, . . . which rest at their extremities on the groins AE, DE. We can estimate the stability of any one of these arches by principles previously established, and find the resultant pressure that it exercises upon the groin. The latter supports a similar pressure from each side; the resultant of these two pressures, which is generally oblique, can then be decomposed into horizontal and vertical components, which are the forces to be used, in their proper positions, in ascertaining the stability of the simple arch constituting the groin, and also of the abutment against which it leans.

In this example the dimensions are given in meters, though any unit may be taken. Let $\overline{AD} = 7.54$; the arc AFD in plan, a semicircle whose radius is thus,

$\overline{OB} = 3.77$; depth of keystone $\overline{KL} = 0.47$; the height of surcharge above it, $LM = 1.26$. Divide the semi-groin AE into a number of equal parts, four in the figure, and suppose each simple arch, as AID, to terminate at the middle, a_4, of its corresponding division. Project up a_1, a_2, a_3, a_4 to b_1, b_2, b_3, b_4, in elevation. Then on this supposition the weight AIF sustained at a_4, is represented in elevation by MNb_4K, supposing the joints vertical. Similarly for arch a_3JI, etc. If an elementary arch a_3JIo could change shape under stress independently of the adjacent arches, its resistance line would be uniquely determined by the theory of the elastic arch, and the centres of pressure for the successive arches b_1, b_2, b_3, b_4, at their respective springings, could be found. As this hypothesis cannot be said to obtain, the points b are indeterminate. It is perhaps safe to take them on or near the centre line down to the vertical through B, which limits the construction. In this example another plan (not so

good) was adopted. Thus, for arch a_4IFe, the centre of pressure at the crown was taken at the upper middle third limit, and at b_4 at the lower middle third limit, from which the resultant at (a_4 b_4) can be found. For the other arcs, as a_3IJ, retain the same value aud position of Q at the crown. We thus find from the diagram for arc AID the resultants in amount, position and direction at the points (a_3, b_3), (a_2, b_2), (a_1, b_1) of the groins, due to all the arcs in the space AEF.

In the following table of volumes and centres of gravity corresponding to MN4K v = volume of trapezoidal prism lying just to right of joint to which it refers = width × mean height × $\overline{\text{IF}}$.

In this case IF = JI = 0.94. l is the distance of the centre of gravity of the trapezoid from the crown, and m the corresponding moment. V is the volume from the crown to the joint to which it refers, found by adding the numbers in column v. Similarly M is formed from

m, and the quotients $\frac{M}{V} = C$, give the distances of the centres of gravity of these volumes V from the crown.

Joint	v	l	m	V	M	C
1	.78	.24	.1872	.78	.1872	.24
2	1.68	.96	1.6128	2.46	1.8000	.73
3	2.04	1.91	3.8964	4.50	5.6964	1.27
4	2.74	2.87	7.8638	7.24	13.5602	1.87
	7.24		13.5602			

Laying off $1^m.87$ from the crown to the left on Q prolonged and drawing from this point a straight line to b_4, we have the direction of the resultant at b_4. Its amount is found by laying off $P_4 = 7.24$ on the vertical, through the point 1.87 to left of the crown, downwards, and then drawing a horizontal line to the resultant, which may now, as well as Q, be scaled off. We thus find $Q = 5.45$.

Next, combining the forces at a_1, a_2, a_3, and a_4, due to the arcs on either side of the groin AE, we have for the vertical components of the resultants at a_1, a_2, a_3, a_4, 1.56, 4.92, 9., 14.48 respectively, being double the numbers given in column V above.

The horizontal component at each point is $Q\sqrt{2} = 7.7$.*

It is evident that the greater the number of divisions IF, JI, &c., the more accurate the result. It is well to test the above volumes analytically.

FORMULAS FOR VOLUME.

63. Let $OB = OK = r$, $KM = c$, and the variable distance $Ed = a_4d = x$. An arc a_4de in plan is shown in elevation by 4321K, where $44^1 = x$ and $04^1 = \sqrt{r^2 - x^2}$.

Draw a tangent at K to intersection H with N4. The area between this tangent, H4 and arc

* If the centres of pressure are all assumed to lie on the centre line (as suggested above) Q must be determined separately for each elementary arch, as just explained for a_4FIe.

4K is easily found by subtracting from the rectangle rx, the triangle $\frac{1}{2}x\sqrt{r^2-x^2}$ and the sector $04K = \frac{r^2}{2}\sin.^{-1}\frac{x}{r}$, so that the area of KMN4 is

$$cx + rx - \frac{1}{2}x\sqrt{r^2 - x^2} - \frac{r^2}{2}\sin.^{-1}\frac{x}{r}.$$

On multiplying this by dx, we find the volume of an elementary arch, parallel to AD, of thickness dx and at a distance x from E. The integral between $x = r$ and $x = o$ gives the total volume of the solid AEF or $\frac{1}{8}$ the groined arch; equal to,

$$\frac{cr^2}{2} + \left(\frac{5}{6} - \frac{\pi}{4}\right)r^3 = \frac{cr^2}{2} + 0.0479\,r^3.$$

As $r = 3.77$, $c = 0.47 + 1.26 = 1.73$ $\therefore$ vol. AEF $= 14.85$. From the Table this volume $=$ $(.78 + 2.46 + 4.50 + 7.24) = 14.98$.*

ARCH OF GROIN AND ABUTMENT.

64. It will conduce to clearness to lay off on an elevation of the groin and abutment, Fig. 17, the forces just found,

* For volumes of groined and cloistered arches, see a full discussion in Mr. Leonard Metcalf's paper on the Groined Arch in Transactions Am. Soc. C. E. Vol. 48: also, see Engineering News for August 23, 1900.

directly over a_1, a_2, . . . at the same heights as b_1, b_2, . . . are above the springing, viz., 4.03, 3.76, 3.15 and 2.12.

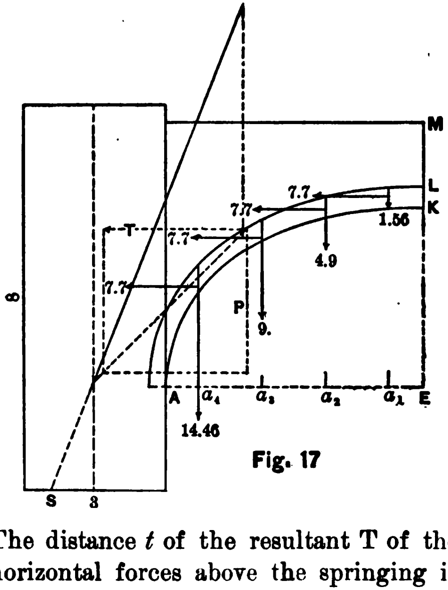

Fig. 17

The distance t of the resultant T of the horizontal forces above the springing is thus,

$$t = \frac{7.7(4.03 + 3.76 + 3.15 + 2.12)}{T = 30.8} = 3.26,$$

from which T can be located. Similarly take moments of the vertical forces about A, to find the distance, p, that their resultant, P, acts to the right of A.

$$\therefore p = \frac{14.46\times.68+9\times2.01+4.9\times3.35+1.56\times4.69}{P = 29.92}$$

$$= 1.73$$

The resultant of T and P passes outside of the arch ring above the springing. On combining it in turn with the weight of the abutment $8\times3\times2$, the final resultant cuts the base at s, $\frac{1}{6}$ the depth from the outer edge.

65. The arch ring of the groin in the actual example, has a depth of $0^m.94$; being double that of the ring as drawn, which may thus be supposed to represent its middle half.

To test its stability, combine the resultant of the forces 7.7 and 1.56, being the pressure on the joint midway between a_1 and a_2, with the resultant of the next two concurrent forces, 7.7 and 4.9, to

find the resultant on the joint midway between a_2 and a_3; next, combine this last resultant with that of the next two concurrent forces and so on. The final resultant on the springing joint should coincide with the resultant of P and T just found.*

The line of the centres of pressure is thus found to keep very near the centre line down to a_4, below which it passes out of the arch-ring, on the extrados side.

The heavy backing will exert horizontal forces to modify this line of resistance, probably keeping it in the arch ring near the springing; for otherwise the intrados joints about the springing must open; but this cannot happen unless the extrados joints open about a_3. If the backing prevents the latter, the former cannot occur, and the centres of pressure are found somewhere in the middle third.

A horizontal thrust, H, at the crown

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* Where the groin is a distinct arch, the weight of its successive portions should be combined with these forces in the proper order.

of the groin will be exerted only when the resistance line $b_1b_2b_3b_4$ approaches the intrados too nearly. If it is desired that the centre of pressure should fall at a certain point on the vertical through a_4 call the arms of H, T and P about this point h, t and p; then,

$$Hh + Tt - Pp = 0.$$

from which H can be found. It may be assumed to act at the centre of the crown joint.

In all ordinary cases no thrust at the crown is needed and an opening can be made there with safety for light and ventilation. This is done in vaulted coverings of reservoirs and filter beds, which are generally rectangular in plan and consist of groined and cylindrical arches springing from the interior piers, and cloistered arches springing from the abutments, which last are walls surrounding the rectangular space.

66. It is usual in such constructions to place the abutments as in Fig. 18.

The space between them is usually covered with simple arches as ABCD. The horizontal thrusts of the two leaning against one abutment, acting at the crown joints combined into one, Q′ $\sqrt{2}$ acting directly over the centre of the abutment, and in

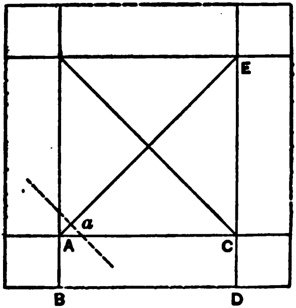

Fig. 18

the direction of a diagonal, as EA. The weights of the two semi-arches acting at their centres of gravity are combined into one, 2P′, acting at *a* on the diagonal AE. On drawing now a section of the abutment along AE produced and laying off the forces Q′ $\sqrt{2}$, 2P′, T, P, H if any,

and the weight of abutment, in their proper positions and combining these forces into one resultant, we ascertain if the centre of pressure at the base of the abutment lies within proper limits. It will be found that the addition of the encompassing arches conduces to stability, the effect of the downward force $2P'$, more than counteracting the effect of the force $Q'\sqrt{2}$.

67. The question of stability has so far alone been considered. As to strength, the simple arch of largest span is to be designed by usual methods and the maximum stress ascertained. If the resultants have been assumed to act at the centre of certain joints, double the uniform intensity of stress there for safety. The same remarks apply to the arch along the groin (if any). Where equal arches spring from a pier in four directions, at right angles to each other, the resultant thrust on such a pier passes through its centre.

For a single vault, as in Fig. 18, the

resultant meets the base of the pier AB on the diagonal EA produced, say at a distance e from its centre. If we call N its normal (vertical) component the maximum and minimum stresses per unit are (7, Art. 12),

$$\left(\frac{1}{d^2} \pm \frac{6\sqrt{2}\,e}{d^3}\right) N$$

where $d = AB$ = side of square base.

The minimum stress is zero when

$$e = \frac{d\sqrt{2}}{12}$$

and for greater values of e, the stress at A is tensile, which should be avoided, as it is best not to count on the tensile stress of the mortar.

Where concrete is used for the arch, the previous investigation holds, though there is certainly decided gain in strength for the arches from the tensile strength of the concrete, both in the arch and backing.*

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* See Transactions Am. Soc. C. E., Vol. 43, p. 65.

Scheffler* has deduced approximate formulas, from which is found (for the case shown by Fig. 18 and a safety factor $n = 3$, by which the horizontal thrust is multiplied), that a groined arch should have the same size of abutment as a cylindrical arch of the same span. For $n = 1$, the case of limiting stability, it was found that the width AB for the groined arch, was $1\frac{1}{2}$ times that for a cylindrical arch of same span AC. As he assumes the horizontal thrust to act at the top of the keystone and makes a number of rude approximations besides, sometimes on the safe side, at other times otherwise, the results are open to doubt. For $n = 1$, however, they seem to be confirmed by the experiments of Rondelet (see Art. 69).

CLOISTERED ARCH.

68. In the cloistered arch, shown in plan, in Fig. 19, ABCD is a square lying in a horizontal plane, whilst EF is a sim-

* "Théorie des Voutes," §57.

ple arch of span EF, and rise equal to the height of the crown at G above the springing. AD and BC are the groins, forming the re-entrant angles on which the smaller arcs, as *ab*, *be*, *cd*, *fg*, etc., meet

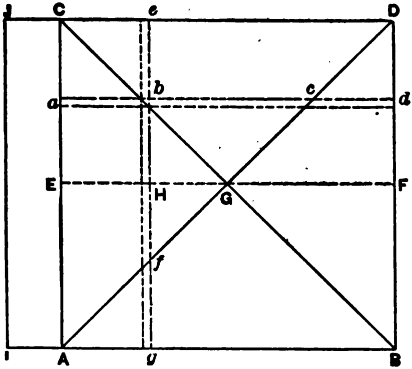

Fig. 19

with an inclined tangent. Thus *ab* is precisely similar in form to the part EH of the full centre arch EF. The elements *bc* or *fb* are thus horizontal. Now the thrust at the crown G of the simple arch EF of small width, is horizontal,

and is computed as for a simple arch. The arcs *ab* and *cd* sustain at *b* and *c* horizontal thrusts communicated through the horizontal element *bc*. When the centres are struck, the tendency to fall causes pressure on the voussoirs in four directions, $\perp$ and $\parallel$ AB, so that the elements, as *bc* and *bf* of the cylinders sustain a uniform horizontal compression in the directions *bc* and *bf*, and the voussoirs composing these elements sustain likewise an inclined thrust (except at the groins, where it is horizontal), in a direction perpendicular to the elements, whose amount is easily determined by the methods affecting simple arches.

69. Thus divide EG into any number of equal parts, and find by usual methods the weights and the positions of their centres of gravity, from the springing AC to any joint, in place of from G to the joint, as hitherto. Part of the table made out then directly applies to each partial arc, as *ab*. On the elevation of the semi-arch EG and of each partial

arch, as *ab*, construct resistance lines, lying as near the centre line as possible, down to the "joint of rupture" (Art. 43). In doing this, proceed as shown in the latter part of Art. 61, to find the horizontal thrust and the resultant at the abutment. With the tables made out as above, the resultant at the abutment must be combined, in turn, with the weights from the abutment to the joint considered, to find the centres of pressure on those joints.

We thus find the various horizontal thrusts, acting at the groins CG and AG in a direction $\perp$AC. On multiplying each of these thrusts by its vertical distance above the springing, and dividing the result by the sum of the thrusts, we find the vertical distance above the springing at which the resultant of the horizontal thrust T, of the part AGC, acts. Similarly, find the horizontal distance to the resultant of the vertical forces, P acting on the part AGC; this resultant representing the weight of

AGC. On combining these resultants, T and P acting in the vertical plane EG with the weight of the abutment, as shown in Fig. 17, we ascertain whether the centre of pressure on the base of the abutment falls within proper limits. Since the arc EF (Fig. 19) causes the greatest thrust, unless the abutment is made to act as one piece, as supposed above, its width should vary, being greatest at E and diminishing to naught, theoretically, at C; the intermediate widths being found in the usual manner from the thrusts of the partial arches resting there. If the curve limiting these widths is assumed to be a parabola, which is doubtless safe, then it is easy to show (see Scheffler, §58) that the abutment AIJC ((Fig. 19) of rectangular section, having the same moment as the one of parabolic section, has a width,

$$e = 0.7303e' = \tfrac{3}{4}e' \text{ (say)}$$

where e' is the width of abutment at E

for the simple arch EF, which can be found by usual methods.

In view of the indetermination existing as to the true curves of resistance of arches such as EF, *ab*, the practical solution is suggested of designing EF as for a simple arch, and giving the same radial depth of arch ring to all the other arches. The width of abutment is then to be found by the last formula.

This last formula is agreeable to the experiments of Rondelet on models; that *for limiting equilibrium and for the same span, the widths of the abutments of domes, cloistered arches, cylindrical arches, and groined arches are as the numbers*, 1, 3, 4 *and* 6.

CHAPTER IV.

DOMES OF MASONRY.

70. The soffit of the dome will be supposed to be generated by revolving a curve about the vertical line representing the rise of the arch called the axis, so

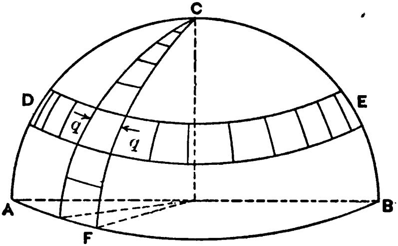

Fig. 20

that every horizontal section of the soffit is a circle. The extrados may be generated by revolving a similar curve or any other figure about the axis similarly for the upper limit of the backing. If we

pass two meridian planes, making a small angle, ψ, with each other, through the axis, we cut from the dome and backing, if any, a solid FC, Fig. 20, being a part of a wedge-shaped figure whose soffit is a portion of a *lune*, when the generating curve is an arc of a circle. This solid, for want of a better name, we shall call a *lune solid.*

Now pass conical joints, perpendicular to the soffit, at certain distances apart; the part of the dome proper, as DE, lying between any two conical joints, will be called a *crown.*

It is proper first to examine the conditions of equilibrium of such a crown; which can moreover form the superior part of a dome open above.

There are developed in these crowns horizontal pressures q, q, whose directions are normal to the joints of the crown, and more intense in the upper than in the lower crowns.

When we afterwards consider the lune solid, CF, limited by meridian planes, it

is necessary to combine the two forces q, q into a single horizontal force Q, acting outwards. It is necessary in all cases that the horizontal thrust at the upper joint may be null.

This is evident for an open dome; for the dome closed at top, which is only a particular case of the open dome, it ~~se-~~ re-sults from the fact, that the surface of the joint at the summit reduces to a line, which cannot support a finite pressure.*

71. Let Fig. 21 represent a lune solid of the open dome considered, and let P_1, P_2, P_3, P_4, laid off in order on the vertical line P_1P_4, represent the weights of voussoirs 1, 2, 3, 4, respectively, with their loads. Let us assume, for the present, that the forces q, q, of the preceding figure act at the centres of the voussoirs; so that the forces Q_1, Q_2, . . ,

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* Dr. Scheffler, in his "Theorie der Gewolbe" (1857,) also, a French translation, "Théorie des Voutes," states that previous authors (Navier, Rondelet) have treated the lune solids as simple arches. I infer that Dr. Scheffler himself is the first who has introduced the forces q above and given a more rational theory.

act through the centres of their corresponding voussoirs, 1, 2, . . . , and horizontally to the left in Fig. 21.

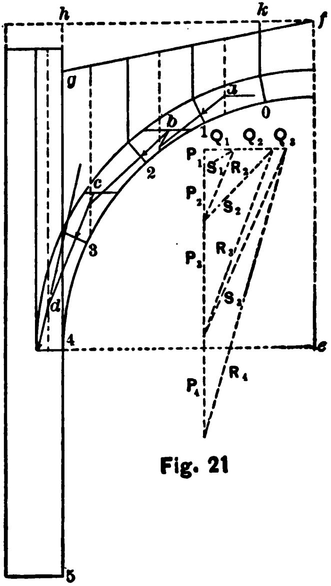


Fig. 21

Now the horizontal thrust at joint o is null. The weight P_1 of the first voussoir

and load, acts through *a* and does not meet joint 1, so that there is no stability unless the "crown" including this voussoir, in tending to fall, exerts a horizontal pressure. The resultant Q_1 of the pressures exerted on both sides of voussoir 1 should be so great that when combined with P_1, the resultant shall cut the joint to which it refers, and make with the normal to this joint an angle not less than the angle of friction. These two conditions hold for every joint. If no joints open the resultants will lie in the middle third. Therefore, if Q_1 be made so large that a line drawn through $a \parallel S_1$ the resultant of P_1 and Q_1, satisfies the above conditions, the point where it cuts joint 1 may be regarded as a *possible* centre of pressure.

If the above conditions are not satisfied for an assumed value of Q_1, Q_1 must be increased.

Now extend the line through *a*, just drawn, to intersection with the vertical through the centre of gravity of the sec-

ond voussoir and load, whose weight is P_2; from this point draw a parallel to R_2, the resultant of S_1 and P_2, and extend it upwards to intersection b, with the horizontal through the centre of the second voussoir along which Q_2 acts. On drawing through b a line to some point on joint 2; a parallel to it, in the force diagram, gives S_2, and cuts off Q_2, as shown in the figure. As before, if S_2 does not make an angle with the normal to joint 2 less than the angle of friction, Q_2 must be increased and the line through b made parallel to S_2 thus found. Similarly we proceed for other joints, until finally we get to a joint, as 3, below which no more forces of the type Q are needed to prevent the resultants on succeeding joints from falling *below* certain limits. The part of the lune solid below joint 3, called the "*joint of rupture*," thus acts as any simple arch; therefore we determine the resultant on joint 4 by combining S_3 and P_4—*i. e.*, by drawing through d a parallel to R_4 of the force diagram,

the resultant on joint 4, to intersection with that joint.

Similarly, if we combine the resultant S_3 on joint 3 with the weight of the entire abutment, we find the centre of pressure on joint 5, which should lie within the middle third.

72. It is evident from the foregoing that an infinite number of equilibrium polygons can be drawn, all satisfying the conditions stated, so that it seems impossible to select the true one. Some of the indetermination, however, can be removed, by considering the elastic yielding of the dome. Thus, since the voussoirs are compressible, and if, as is usual, the actual resultant on the springing joint passes to the left of the centre, the outer edge is most compressed, and to allow this the haunches must spread and the top of the arch descend, so that about joint 3, the centre of pressure passes below and at the top, above the centre line. This is all the more evident if the springing joint opens at the inner edge. This

view will be illustrated by an example in Art. 82.

As a modification of the above hypothesis, we may assume that the resultants S_1, S_2, . . . , are tangent to the centre line, from the crown to the joint of rupture. It will be found that this involves raising Q_1, Q_2, ... slightly above the centres of the voussoirs. The construction is much simplified by this assumption, which will be illustrated more fully in Art. 77.

73. From the definition of Art. 70, and a plan of a voussoir bounded by the two meridian planes whose included angle in arc is ψ, and which is solicited by the two horizontal forces q, q, (acting perpendicular to its vertical faces), whose resultant is Q, we have,

$$\tfrac{1}{2}Q = q \sin. \tfrac{1}{2}\psi.$$

If a = half span, and ε = horizontal width of voussoir at the springing, then $a\psi = \varepsilon$. When the angle ψ is small, i.e.,

when ε is made small enough, we have from the above equation,

$$q = \frac{Q}{\psi} = \frac{a}{\varepsilon} Q ;$$

from which q_1, q_2 . . . can be computed, as soon as Q_1, Q_2, are found by the construction above.

NUMERICAL EXAMPLE.

74. Let us take the half span (Fig. 21) equal to 9.42 units, the depth of arch ring 0.94; and let the inclined line fg limit the load, the point f being 12.24 above the centre e of the soffit, and g, 1.98 lower than f. The radius $fk = 1.86$. Now divide the horizontal hk into six parts, each 1.26 wide, in place of three as before; drop verticals through the points of division, and from their intersection with the extrados draw the joints 0 to 6. We shall suppose approximately that the figures so formed are trapezoids, whose area equals the mean height multiplied by the width.

The volume of a voussoir and load can be exactly determined by the theorem of Pappus:

The volume of a solid of revolution is equal to the product of the area of the generating surface

and the distance described by its centre of gravity while generating the body.

Let x = distance from axis ef to the medial vertical of a trapezoid,

$\therefore$ volume of trapezoidal solid =

$$\mathrm{A}x\psi = \mathrm{A}\,\frac{x}{a}\,\varepsilon$$

where A = area of trapezoid and a = radius $e4$.

Take $\varepsilon = 1$ for simplicity. The thickness of voussoir at the springing = ε selected is immaterial except in finding q, since multiplying the weights of voussoirs and abutments by the same quantity does not change their ratios.

In the following table, column (1) refers to joint, column (2) gives the height of the trapezoidal solid, column (3) its width, column (4) its thickness = $\frac{x}{a}$, and column (5) their product representing the forces P_1, P_2, , . .

(1)	(2)	(3)	(4)		(5)
1	2.65	1.26	.25	P_1	0.83
2	2.83	1.26	.38	P_2	1.35
3	3.23	1.26	.50	P_3	2.03
4	3.92	1.26	.63	P_4	3.11
5	5.03	1.26	.76	P_5	4.82
6	7.05	1.26	.89	P_6	7.90
7	10.99	0.94	1.00	P_7	10.33

The construction is now proceeded with exactly as described for Fig. 21, which is in fact a drawing to these dimensions.

The induced forces Q_1, Q_2, . . . , were conceived to pass through the centres of the voussoirs. The resultants S_1, S_2, . . . , on the joints were made first to pass through the lower middle third limits, and afterwards through the centres of those joints. In both cases the joint marked 3 in Fig. 21, was the joint of rupture; the resultant on the springing joint, in the first instance, coinciding with the resultant as drawn in Fig. 21; in the last case passing nearly through the extrados. The total horizontal thrust in the first case = 7.75; in the last, 8.13. If we take the width of abutment at 3, its height above the springing 10.99, its depth below it 7.01; its mean thickness is $\frac{10.92}{9.9} = 1.1$, and its total volume, including a part of the arch is $3 \times 18 \times 1.1 = 59.4$; which combined with the resultant S_6 on the joint marked 3 in Fig. 21, cuts the base, for the first case noted above, only 0.05 outside of the middle third, in the last case 0.18 outside.

The force $Q_2 = 2.3$ is the largest of the forces Q_1, Q_2 . . . , whence by Art. 94,

$$q = \frac{a}{e} Q_2 = 9.9 \times 2.3 = 22.77 \text{ cubic units}$$

of stone. The force q acts on an area of 1.23 square units. There is evidently no danger of crushing from the horizontal thrust around the second crown from the top, as stone will bear on a square unit a pillar of a square unit section and several thousand units high. Similarly for resultants on all the joints. We conclude that with the backing used and by increasing the depth of arch ring at the springing about one-third, the arch will be stable.

FORMULAS FOR VOLUME.

75. When the soffit and exterior surface of the dome are both surfaces of spheres having the same centre, the volumes of the voussoirs are easily obtained. Thus, by geometry, the area of the zone formed by revolving an arc as rs (Fig. 22) about the axis ac is equal to the altitude h of rs multiplied by $2\pi r$, r being the radius $\overline{as}$ of the sphere. Pass now two meridian planes through ac, whose included angle is $\frac{1}{n}$ of a circumference. The part of the zone included between them has an area $\frac{2\pi rh}{n}$ so that the pyra-

mid formed on this base with a vertex at a, has a volume $\frac{2\pi r^2 h}{3n}$.

Similarly the pyramid having the part of the zone represented by tv as a base, has a volume, $\frac{2\pi r'^2 h'}{3n}$ where r' = radius at and h' = altitude of arc tv. Therefore the volume of the voussoir $rstv$ included between the meridian planes and the conical joints rv and st is,

$$V = \frac{2\pi}{3n}(r^2 h - r'^2 h')$$

where r and h are the radius and altitude of the exterior arc, r' and h' of the interior. If the altitudes of the type h are made equal in successive arcs, the values h' will all be equal. This can be shown as follows: Draw horizontal lines through r and v, Fig. 22, and drop verticals h and h' from s and t to them, and call the chords rs and vt, c and c', $\therefore$ from similar triangles,

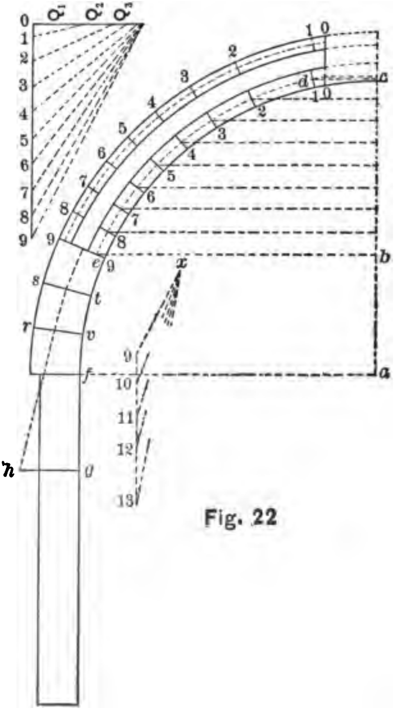
$$\frac{h}{h'} = \frac{c}{c'} = \frac{r}{r'} \therefore h' = \frac{r'}{r} h.$$

$\therefore$ When h is constant for successive arcs of the extrados, h' is similarly constant for the corresponding arcs of the intrados. Similarly the centre line of the arch ring will be divided into arcs of the same altitude when h is constant—and conversely.

It follows, that if *de*, Fig. 22, represents the centre line of part of a lune solid, having exterior and interior radii r and r', if d is projected on the axis at c and e at b; also if bc is divided into equal parts, and horizontals are drawn through the points of division to intersection with arc *de*, the normal joints being passed through these intersections, then by the formula above, the voussoirs so formed are all equal in volume.

76. Let us refer again to the inner dome *de* (Fig. 22).

If we pass horizontal planes midway between the horizontals first drawn, also pass conical joints through their intersec-



0
1
2
3
4
5
6
7
8
9
Q1
Q2
Q3
1 0
2
3
4
5
6
7
8
9
d
c
1 0
2
3
4
5
6
7
8
e 9
b
x
8
t
r
v
9
10
11
12
13
f
a
h
g

Fig. 22

tion with the centre line *de*, we divide the previous voussoirs exactly in half, so that *the centres of gravity* of the first voussoirs lie on these supposed intermediate conical joints. They also lie nearly on the centre line *de*, and the error of so regarding them can be made as small as we choose by sufficiently diminishing ψ, the angle included between the two meridian planes and the height of the voussoir.

The centres of gravity of the voussoirs will therefore be assumed to be on the centre line of the elevation of the medial meridian section *de*, at the intersections of the horizontals drawn midway between the first horizontals drawn that divide *bc* into equal parts.

DOME OF TWO SHELLS WITH LANTERN.

77. Fig. 22 represents a meridian section of the Church of St. Peter at Rome. The dimensions given by Scheffler, as I understand them,* are as follows :

* I have used the dimensions given in his Tables, which differ, in some respects, from those in the text.

The radius of the soffit is 72 feet, and of the outer surface, $\overline{as} = 83.8$. At 31 feet above the springing, the structure is composed of two domes, the outer having a thickness of 2.6 feet, the inner being 4.1 thick at d and 5.1 at e, so that the centre line de is described from a centre slightly below a on the vertical $\overline{ac}$ produced. The dome has an opening at top 12.4 radius, and the lantern supported at the top is equivalent in weight to a block of stone 2.1×56.6, of which the outer shell supports one-third and the inner two-thirds. The first voussoir at the top, in both shells, is made 2.1, horizontal width; the altitude of the centre line, $\overline{bc}$, for the part de, is then divided into 8 equal parts and the joints drawn as in the figure, a similar construction applying to the outer shell. The part below the two shells is similarly divided into 3 equal parts. Applying the formula just deduced in Art. 75, measuring the altitudes on a drawing to a scale of 4 feet to the inch, we find the volumes of the voussoirs

like *rstv*, $\frac{2\pi}{3n}$ 28696, the voussoirs of the outer shell, $\frac{2\pi}{3n}$ 3957, except the top one, whose volume is $\frac{2\pi}{3n}$ 359 cubic feet. The volume of the top voussoir of the inner shell is $\frac{2\pi}{3n}$ 501 cubic feet. The voussoirs 2 to 9 of the inner shell, were each, in turn, assumed to have an outer surface concentric with the soffit, of radius equal to the mean radius of the outer surface for the voussoir considered, i.e., equal to 72 + mean thickness in feet of voussoir. We thus find the volumes of voussoirs 2 to 9 equal to the constant $\frac{2\pi}{3n}$ multiplied in turn by 4695, 4805, 4915, 5000, 5124, 5206, 5267 and 5400.

The volume of the lantern, by the law of Pappus (Art. 74) $= 2.1 \times 56.6 \times \frac{2\pi \times 13.45}{n} = \frac{2\pi}{3n}$ 4794, one-third of which is added to the volume of voussoir 1 of

outer shell and two-thirds to that of voussoir 1 of the inner shell. The part *fgh*, 10.3 wide and 23.7 high, has a volume,

$$10.3 \times 23.7 \times \frac{2\pi \times 77.15}{n} = \frac{2\pi}{3n} 56500.$$

This part has not the full width of the bottom voussoirs as drawn in the figure.

TENSION IN IRON BANDS. THRUSTS.

78. We now lay off on vertical lines the weights just found, omitting the common constant $\frac{2\pi}{3n}$.

The loads affecting the outer shell are laid off to its left; those pertaining to the inner shell are not shown in the figure.

To be on the safe side, we shall assume that the resultants on the joints from the summit to the joint of rupture are tangent to the centre line of the ring. Thus for the outer shell, draw through the points 1, 2, . . . , of the force line, parallels to tangents to the centre line at

joints 1, 2, . . . (or ⊥ to radii). These lines cut off successive distances on the horizontal through o, equal to the radial forces Q_1, Q_2, . . . exerted by the successive crowns 1, 2, . . .

We find that below joint 6 there is no longer a radial force needed; so that below that joint the curve of resistance is continued to joint 9 as in a simple arch.

Similar results were found for the inner shell. The centres of pressure on joints 9 of the outer and inner shells are at the outer middle third limit for the outer shell, and slightly above the centre line for the inner shell. This necessitates spreading about joints 6, so that the centre of pressure there is below the centre line, hence the actual horizontal thrust is less than estimated, as stated above.

Now combining the resultants at joint 9 into one, its position being found by moments, and laying off $\overline{x9}$ equal and parallel to it, we continue the line of resistance as per dotted line to joint hg. The suc-

cessive volumes of voussoirs are laid off on the force line 9 . . . 13. This second force diagram is drawn to a smaller scale than the preceding.

79. At joint hg the centre of pressure passes outside the joint, so that the dome cannot be regarded as sufficiently stable in itself. If rotation occurs, the line of resistance would approach the extrados at the summit, the intrados at the joints of rupture, and the extrados at joint hg.

By encircling the dome just above or below the springing by a band of iron of sufficient cross section, stability may be assured.

The total horizontal thrust of the lune solid (being $\frac{2\pi}{3n} \times$ the horizontal component of $\overline{x9}$) is

$$Q = 39600 \frac{2\pi}{3n} = 13200\psi$$

cubic feet of stone $= 924\psi$ tons, if we put the weight of a cubic foot of stone at 157 lbs. $= .07$ ton.

If this is to be entirely destroyed by the iron band, so that the resultants below the springing will all be vertical, we have the tension in the band by Art. 73, when ψ is small,

$$q - \frac{Q}{\psi} = 924 \text{ tons.}$$

Now iron, exposed to a dead stress alone, may safely be subjected to a stress of $7\frac{1}{2}$ tons per square inch; so that the bar may have a cross section of 123 square inches.

If the iron stretches $\frac{1}{12000}$ of its length for every ton per square inch, the ring whose diameter is 168 feet will elongate 0.33 feet, so that the diameter is increased 0.04 feet. There will consequently be a slight deformation of the arch; in consequence, the top of the abutment moves slightly outwards, and the pressure on its base is not generally vertical; i.e., the iron band has not totally destroyed the horizontal thrust.

A similar effect follows from a rise of

temperature, the reverse for a fall; for the temperature of the iron will suffer greater deviations from the mean than the masonry. This result is modified again by the different coefficients of expansion of stone and iron; granite and marble expanding less than iron, sandstone, more.

There is no necessity in the voussoir dome for additional hoops below the first, unless the first is too small to destroy the outward thrust. The problem is then really indeterminate of ascertaining the precise amounts of the stresses sustained by the several hoops. The one nearest the joint of rupture of course will sustain by far the greatest part; the hoops, at joints where no spreading would occur, if they were not applied, not sustaining any. It would seem best to make these hoops of steel as it does not stretch as much as iron. Wire cables with a means of tightening would be especially convenient.

80. We see that the thrust of the type Q is greatest on voussoirs 1 of both shells.

Thus for outer shell,

$$Q_1 = 16900\frac{2\pi}{3n} = 5633\,\psi\,;$$

and for the inner shell, Q_1 $6000\,\psi$, which multiplied by .07 give the thrusts in tons. By Art. 73, we have $q = \frac{Q_1}{\psi}$ or 394 tons and 420 tons respectively. Now voussoir 1 of outer shell has a lateral area of 6 square feet; the lower voussoir 1, an area of 10.45 square feet, so that the pressures per square foot are 66 and 40 tons respectively, which good stone can stand easily.

The thrusts Q_1 of both shells is probably less than assumed, for the force diagrams indicate a small value for Q_2—in fact, for the lower shell Q_2 nearly vanishes—but the compression around the ring of the first crown would necessarily bring the second more in action thereby increasing its thrust. The tendency then is to equalize more nearly the values of the

thrusts Q_1, Q_2, . . . than as given by our construction.

81. It is evident that the greatest economy is subserved, by the employment of one or more *thin* domes to a short distance below the joint of rupture, as is the practice generally in large domes. One shell would suffice if the weight of lantern (if any) could be carried by it; otherwise two or more should be used.

The abutment below joint *hg* is counterforted so as to present a greater width than shown in the figure. The introduction of the hoop of course prevents any movement in it, so that the stability of the whole structure is assured.

DOME WITHOUT LANTERN. — SUGGESTION.

82. We shall give now the spherical dome closed at top to illustrate the other view taken in Art. 72 of the position of the line of resistance, besides other points not noticed before. This dome, Fig. 23,

has a thickness of one-fifteenth the span, and is supposed to have immovable abutments.

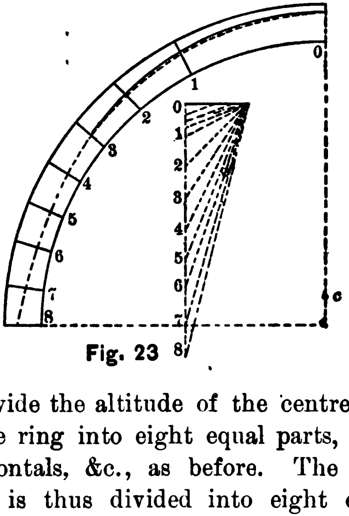


Fig. 23

Divide the altitude of the centre line of the ring into eight equal parts, draw horizontals, &c., as before. The lune solid is thus divided into eight equal parts which lay off on the force line 0 . . . 8.

Now a dome of this kind only *fails* by rotating about the outer edges of joints at the crown and abutment, and the

inner edges about joint 4; each lune solid separating from the others and acting as a simple arch. It cannot fail by the haunches falling in.

For a dome then of small stability, the line of the centres of pressures passes nearly through the top, the intrados edge at joint 4, and the outer edge at joint 8.

For a dome of greater stability its position depends upon the amount of spreading at the haunches, and the consequent rocking at joint 8. If a line of resistance can just be inscribed in certain limits, equally distant from the centre, then it is probable, from a consideration of the way in which an arch settles, that the actual resistance line nearly touches these limiting curves towards the extrados side at the top and abutment, and next the intrados side at the joint of rupture.

For example, take the middle third limits and draw an arc of a circle through the upper limit at the summit and the lower limit at joint 4 with a centre *c* and

assume that this arc coincides with the line of resistance a certain distance from the top, the resultants being tangent to it.

Then at some joint as 2 continue the resistance line down to the springing with the horizontal thrust found at 2. If the line so found does not keep within the middle third, let it be commenced at another joint, until one is found that will satisfy the conditions if possible. On dividing voussoir 2 into four others, it was found that a line of resistance, continued as for a simple arch, from where the arc above cuts the upper joint of voussoir $1\frac{3}{4}$, keeps almost entirely in the middle third, as shown by the dotted line: cutting joints 4 and 5 nearly at the lower limits and joint 8 at the onter limit.

83. In case no such line can be found, the radial depth of arch ring must be increased until the requirement obtains. Where only one such line can be found, the true centres of pressure are doubtless slightly outside the limiting curves at

joints 4 and 8, as in the case of a cylindrical arch.*

The simple arch is found to begin slightly below joint 1 when the upper middle third limit is assumed as the resistance line above it. Where the centre line of the arch ring is assumed for the upper curve, the horizontal thrust becomes constant, and simple arch action begins lower down, between joints 3 and 4 (Art. 78), so that much less of the dome acts as a simple arch in this case.

The true resistance line, even for immovable abutments, depends upon the cutting of the stones as well as the elastic yielding of the dome, and probably lies between the two given by the above construction and that of Art. 78. If domes are designed then on both hypotheses, so that the resistance line can just be contained within the middle third, a mean

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* See Prof. Eddy's "New Constructions in Graphical Statics," p. 56, for a direct method of locating the resistance line to satisfy certain conditions.

of the results should give a good working value to the thickness.

In the example above, if we divide voussoir 1 into four equal ones, we find that the circumferential thrusts *per square unit* on each crown going from the top are proportional to 2.5, 2.3, 1.9, and 1.8 respectively, so that the unit stresses decrease going from the summit.

With a heavy *lantern* a very small crown of voussoirs next the summit exerts a comparatively large thrust; so that the upper crown is compressed sufficiently to bring the next crown more in action, and so on down. In this case, the point *o* of the force line (Fig. 23) is moved upwards, and since the inclined thrusts above the joint of rupture do not change direction, the joint of rupture is much higher than when there is no lantern. The reverse happens when weight is removed from the top, as in the case of the open dome, so that the lantern with the open dome may have the joint of rupture near the usual position.

CONCRETE DOME.

84. The case of the *concrete dome* is nearly similar to that of the metal one. Rankine, in his "Applied Mechanics," p. 267, has given formulas for immediate use in practice, for both the spherical and conical domes.

Graphically, the joint of rupture is obtained as in Fig. 22 (Art. 75); then on continuing to draw resultants perpendicular to the radii at joints 5, 6, 7, . . . , through the points 5, 6, 7, . . . , of the load line, they will be found to cut off less and less distances on the horizontal through o. The total horizontal thrust diminishes to zero at the abutment, where the tangent is vertical. The forces of the type Q act on the voussoir outwards and thus give rise to tensile circumferential stresses of the value $q = Q \div \psi$ (Art. 73).

If the concrete is not strong enough to resist the full amount of this tension, it can readily be reinforced by steel chains

or cables enclosed in the successive "crowns" below the joint of rupture.

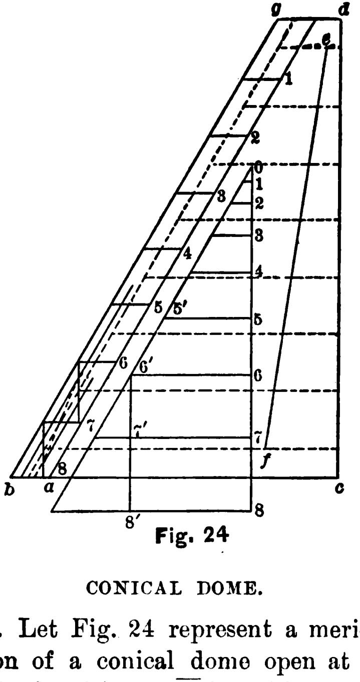

Fig. 24

CONICAL DOME.

85. Let Fig. 24 represent a meridian section of a conical dome open at top. Divide the altitude $\overline{cd}$ into (eight) equal

parts, and pass horizontal planes through the points of division giving the joints 1, 2, . . . Midway between these joints draw the horizontal radii of the centre line shown by the dotted lines. If we call the horizontal thickness of the ring $ab = t$, the vertical distance between any two joints h, and the mean radius of the centre line between these joints by r, we have for the volume of the voussoir included between the two joints and two meridian planes, making an angle ψ with each other,

$$V = htr\psi,$$

according to the law of Pappus.

The weights of the successive voussoirs vary therefore as r. Therefore lay off on the vertieal force line, 0 . . . 8 successive distances, proportional to the dotted radii, beginning with the first voussoir. The line $\overline{ef}$ cuts off one-fourth of these radii counting from $\overline{cd}$.

Next draw the line $\overline{06'} \parallel \overline{bg}$ and pass horizontals through the points 1, 2, . . ,

of the force line. If the resultants on the joints are assumed to coincide with, or be parallel to the centre line, the hypothenuses of the triangles just formed represent the stresses on the joints. Thus $\overline{06'}$ is proportional to the stress on joint 6, and $\overline{66'}-\overline{55'}=$ horizontal radial force Q exerted by the sixth ring from which the ring compression $q = Q \div \psi$ can be found. Similarly for the other rings or crowns.

In order that the centres of pressure fall on the centre line all the way down to the base, it is necessary that the drum on which the base rests can resist the total horizontal radial thrust at the base, as the resultant there must act parallel to *bg* and have a vertical component equal to the weight of the lune solid considered.

86. The least horizontal thrust, consistent with no joints opening, may be found as follows :

Assume the centres of gravity of the voussoirs to lie on the centre line midway between the joints. Then combine the

thrust at joint 6, $\overline{06'}$ (as found above), supposed to act at the exterior middle third limit, with the weight of voussoirs below it; if the resultant, $\overline{08'}$, strikes the base at the *inner* limit, the horizontal thrust, $66' = 88'$, is a minimum in order that no joints open. In the above figure it was found, on a second trial, that at joint $6\frac{1}{4}$ the horizontal thrust first became constant, so that the part of the dome below it exerted no cirumferential thrust.* The part below joint $6\frac{1}{4}$ thus acts as a simple arch. If the drum gives slightly the true thrust is doubtless found between the two limits. If the drum cannot resist the horizontal thrust proportional to $66'$, the resistance line will approach the extrados just above joint 6 and the intrados at the base; at the same time the lowest "crown" spreads and meridian joints open at the base if it cannot resist tension. The horizontal radial thrust is

* In Prof. H. T. Eddy's "New Constructions in Graphical Statics," a direct method is substituted for this tentative one.

now less than 66′ and becomes less as the drum gives still farther, until finally the resistance line nearly touches the contour lines and the lower part of the dome, acting as a series of isolated arches, falls, followed by the upper portion.

The metal dome, or the one of concrete, is investigated as in Art. 85. It is better, in both cases, that the lower ring should be strong enough to furnish all the tension needed, so that the drum be subjected only to vertical forces.

When the dome is surmounted by a lantern, the proportionate weight is laid off vertically above o, Fig. 24, and the construction is proceeded with as before.